Endogenous Stackelberg Leadership*

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Abstract

We consider a linear quantity setting duopoly game and analyze which of the players will commit when both players have the possibility to do so. To that end, we study a 2-stage game in which each player can either commit to a quantity in stage 1 or wait till stage 2. We show that committing is more risky for the high cost firm and that, consequently, risk dominance considerations, as in Harsanyi and Selten (1988), allow the conclusion that only the low cost firm will choose to commit. Hence, the low cost firm will emerge as the endogenous Stackelberg leader. *Journal of Economic Literature* Classification Numbers: C72, D43.

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1 Introduction

Ever since Von Stackelberg wrote his *Marktform und Gleichgewicht* in 1934, it has been well-known that in many duopoly situations a firm is better off when it acts as a leader than when it acts as a follower. Since each firm will strive to obtain the most favorable position for itself, the question arises which of the two duopolists will gain victory and obtain this leadership position. Von Stackelberg concluded that in general it is not possible to answer this question theoretically (Von Stackelberg, 1934, pp. 18-20). In this paper, we consider the special case of a linear quantity setting duopoly game and show that in this case the role assignment may follow from risk considerations. Specifically, we demonstrate that committing is less risky for a low cost firm so that such a firm will emerge as the Stackelberg leader.

Our work is inspired by an idea of Thomas Schelling. Of course, Schelling is most well-known for his general demonstration of the value of commitment, i.e. that committing is beneficial for a player who is the only one able to make a commitment. Schelling realized that, as a consequence, all players in the game will attempt to commit themselves and that a coordination problem might arise: committing is beneficial only if the opponent does not commit, it might be (very) costly if the opponent also commits himself. This in turn implies that a player might decide not to commit himself since he fears that the opponent might commit as well and since the costs associated with the resulting "Stackelberg war" might be too high (Schelling, 1960, p. 39). Hence, there is a fundamental trade-off between flexibility and commitment. Schelling pointed out this trade-off, but he did not provide a formal analysis of it, he did not solve the game. Our aim in this paper is to provide a full solution for the linear 2-person duopoly game.

We consider a quantity setting duopoly game with linear demand and constant marginal cost. One firm is more efficient, i.e. has lower marginal cost, than the other. The formal model used to analyze the trade-off between commitment and flexibility is the 2-stage action commitment game from Hamilton and Slutsky (1990). The rules are as follows. Each duopolist has to move (i.e. to choose a quantity) in one of two periods; choices are simultaneous, but, if one player chooses to move early while the other moves late, the latter is informed about the first-mover's choice before making his decision. Hence, moving early is profitable if one is the only player to do so, but it is costly if the other commits as well. This timing game has several equilibria, in particular, each of the Stackelberg outcomes of the underlying duopoly game is an equilibrium. As Hamilton and Slutsky pointed out, these are the only pure undominated equilibria of the game. We select the solution of the game by using the risk-dominance concept from Harsanyi and Selten (1988). This concept allows one to quantify the risks involved with the two candidate solutions and, hence, it enables to resolve the trade-offs. Risk considerations show that committing is less risky for the firm that has the lower marginal cost. This safer equilibrium in which the low cost firm moves first is the neutral focal point and, adopting the risk dominance concept, the players will coordinate on it.

Some intuition for this result might be obtained by looking at the 2×2 game in which each player is restricted to use one of two strategies: either to commit himself to his Stackelberg leader quantity or to wait till the second period and then best respond to the quantity chosen by the opponent, with players choosing their Cournot quantities in the second period if neither player moved in the first period. (See Table 1 in Section 3 for the payoff matrix.) Both Stackelberg outcomes appear as strict equilibria in this game and it is well-known that risk dominance allows a simple characterization for such 2×2 games: the equilibrium with the highest (Nash) product of the deviation losses is the risk dominant one. (Harsanyi and Selten, 1988, Lemma 5.4.4). In Section 3 we show that the equilibrium where the low cost firm commits is risk dominant in this reduced game. The intuition is that, if player 1 has higher marginal costs, then his reaction curve is below the reaction curve of player 2, so that his Stackelberg and Nash quantities are closer together, which implies that he can gain less from committing himself than player 2 can. On the other hand, player 1 incurs greater losses than player 2 does if both players commit themselves. As a consequence, player 1 is in a weaker bargaining position to push for his most favored outcome and he will lose the battle.

If the risk-dominance relation between our two candidate solutions could always be decided on the basis of the 2×2 game spanned by them, then our problem could be solved by straightforward computation. Unfortunately, the problem posed in this paper is not that simple to solve and the above mentioned characterization of risk dominance is of limited use for the problem addressed. In our "action commitment" game, a player has infinitely many strategies available; the choice is not simply between committing to the Stackelberg leader quantity and waiting. Furthermore, it is known that, in general, the reduced 2×2 game spanned by the two equilibrium candidates may capture the overall risk situation rather badly. Consequently, to find the solution of the game, there is no recourse but to apply risk dominance to the overall game. Now risk dominance is defined by means of the tracing procedure and the fact that this procedure is rather complex and difficult to handle forces us to restrict ourselves to the linear case. Even in this most simple linear case, the computations are already rather involved, they become very cumbersome in the more general case. Nevertheless, the main result of this paper is that risk dominance indeed selects the equilibrium in which the low cost firm leads.

The present paper is part of a small, but growing, literature that aims at endogenizing the first mover in oligopoly models. Ours is the first paper in which a specific Stackelberg outcome is derived from a model in which the duopolists are in symmetric positions ex ante and in which only endogenous (strategic) uncertainty is present. Related papers either put firms in asymmetric positions to start with, or add exogenous uncertainty (about production costs or market demand), or admit multiple equilibrium outcomes.

Hamilton and Slutsky (1990) consider the same game as we do and they show that the two Stackelberg equilibria are the only pure strategy equilibria in undominated strategies. Hence, they conclude that a Stackelberg outcome will result but they cannot tell which one. Sadanand and Sadanand (1996) analyze the same model when firms face demand uncertainty, which is resolved before production in the second stage. (Also see Sadanand and Green (1991).) There is always a symmetric (Cournot) equilibrium: both firms move late when uncertainty is large and early when there is no uncertainty. In addition, both Stackelberg outcomes can be sustained as equilibria provided that uncertainty is not too large. Hence, to select a unique Stackelberg outcome it is necessary to assume that uncertainty influences the duopolists in an asymmetric way. An interesting asymmetric variant that Sadanand and Sadanand analyze is a large firm versus fringe model. Since each fringe firm individually is too small to influence output, the unique equilibrium now has the large firm committing itself, while the small firms remain flexible. Spencer and Brander (1992) study a similar duopoly model with demand uncertainty. However, they assume that a firm who moves early is informed about the time at which the opponent moves, which simplifies the analysis considerably. For example, when both firms decide to move early, it follows that they will produce Cournot quantities. In a symmetric setting, both firms will move early (resp. late) when uncertainty is low (resp. high), so that in each case a Cournot outcome results. A Stackelberg outcome may result when firms are in asymmetric positions: when one firm is much better informed about the exogenous shock than the other, then the better informed firm may emerge as the Stackelberg leader. A different type of asymmetry is considered in Kambhu (1984): one firm is risk neutral and the other is risk averse. In this case, the risk neutral firm may arise as the Stackelberg leader. Mailath (1993) puts the firms in asymmetric starting positions. One firm is informed about demand, while the other faces uncertainty and only the informed firm has the option to move first. In the unique "intuitive" equilibrium the informed firm indeed acts as a Stackelberg leader, even if it could earn higher ex ante profits by choosing quantities simultaneously with the uninformed firm.

Saloner (1987) considers a model related to the one discussed here in which also two periods of production are allowed. Firms simultaneously choose quantities in the first period; these become common knowledge and then firms simultaneously decide how much more to produce in the second period before the market clears. Saloner shows that any outcome on the outer envelope of the two reaction functions lying inbetween the two Stackelberg outcomes can be sustained as a subgame perfect equilibrium. Ellingsen (1995) notes that only the two Stackelberg outcomes survive iterated elimination of (weakly) dominated strategies in this game. Pal (1991) generalizes Saloner's analysis by allowing for cost differences across periods. If production is cheaper in the first period (resp. much cheaper in the second period), then both firms produce their Cournot quantities in the first (resp. second) period. In the intermediate case, where costs fall slightly over time, either of the two Stackelberg outcomes can be sustained as a subgame perfect equilibrium. Hence, none of these papers can make a selection among the Stackelberg outcomes.

The remainder of this paper is organized as follows. The underlying duopoly game as well as the action commitment game from Hamilton and Slutsky (1990) are described in Section 2, where also relevant notation is introduced. Section 3 describes the specifics of the tracing procedure as it applies in this context and defines the concept of risk dominance. The main results are derived in Section 4. Section 5 concludes. Some proofs are relegated to the Appendix.

2 The Model

The underlying linear quantity-setting duopoly game is as follows. There are two firms, 1 and 2. Firm i produces quantity q_i at a constant marginal cost $c_i \geq 0$. The market price is linear, $p = \max\{0, a - q_1 - q_2\}$. Firms choose quantities simultaneously and the profit of firm i is given by $u_i(q_1, q_2) = (p - c_i)q_i$. We assume that $3c_i - 2c_j \leq a \;\; (i, j \in \{1, 2\}, i \notin j)$, which implies that a Stackelberg follower will not be driven out of the market. We will restrict ourselves to the case where firm 2 is more efficient than firm 1, $c_1 > c_2$. We write $a_i = a - c_i$.

The best reply of player j against the quantity q_i of player i is unique and is given by

$$b_j(q_i) = \max\{0, (a_j - q_i)/2\}. \tag{2.1}$$

The unique maximizer of the function $q_i \mapsto u_i(q_i, b_j(q_i))$ is denoted by q_i^L (firms i's Stackelberg leader quantity). We also write q_j^F for the quantity that j will choose as a

Stackelberg follower, $q_j^F = b_j(q_i^L)$, and $L_i = u_i(q_i^L, q_j^F)$ and $F_i = u_i(q_i^F, q_j^L)$. We write (q_1^N, q_2^N) for the unique Nash equilibrium of the game and denote player i's payoff in this equilibrium by N_i. For later reference we note that

$$q_i^L = \frac{2a_i - a_j}{2}, \quad q_i^N = \frac{2a_i - a_j}{3}, \quad q_i^F = \frac{3a_i - 2a_j}{4}, \tag{2.2}$$

$$L_i = \frac{(2a_i - a_j)^2}{8}, \quad N_i = \frac{(2a_i - a_j)^2}{9}, \quad F_i = \frac{(3a_i - 2a_j)^2}{16}. \tag{2.3}$$

As is well-known

$$q_i^L > q_i^N > q_i^F, \quad (i = 1, 2) \tag{2.4}$$

$$L_i > N_i > F_i, \quad (i = 1, 2) \tag{2.5}$$

hence, each player has an incentive to commit himself.

To investigate which player will dare to commit himself when both players have the opportunity to do so, we make use of the *two-period action commitment game* that was proposed in Hamilton and Slutsky (1990). The rules are as follows. There are two periods and each player has to choose a quantity in exactly one of these periods. Within a period, choices are simultaneous, but, if a player does not choose to move in period 1, then in period 2 this player is informed about which action his opponent chose in period 1. This game has proper subgames at $t = 2$ and our assumptions imply that all of these have unique equilibria. We will analyze the reduced game, g^2, that results when these subgames are replaced by their equilibrium values. Formally, the strategy set of player i in g^2 is $\mathbb{R}_+ \cup \{W_i\}$, where W_i denotes i's strategy to wait till period 2, and the payoff function is given by

$$u_i(q_i, q_j) \quad = \quad (a_i - q_i - q_j)q_i \tag{2.6}$$

$$u_i(q_i, W_j) = (a_i - q_i - b_j(q_i))q_i \tag{2.7}$$

$$u_i(W_i, q_j) = (a_i - q_j)^2/4 \tag{2.8}$$

$$u_i(W_i, W_j) = (2a_i - a_j)^2/9 \tag{2.9}$$

It is easily seen that g^2 has three Nash equilibria in pure strategies: Either each player i commits to his Nash quantity q_i^N in the first period, or one player i commits to his Stackelberg leader quantity q_i^L and the other player waits till the second period. One also notices (with Hamilton and Slutsky (1990)) that the first (Cournot) equilibrium is in weakly dominated strategies (committing to q_i^N is dominated by W_i in g^2), hence, one expects that only the (Stackelberg) equilibria in which players move in different periods are viable. Below we will indeed show that the Cournot equilibrium is risk dominated by both Stackelberg equilibria (Proposition 1). It should be noted that besides these pure equilibria, the game g^2 admits several mixed equilibria as well. These mixed equilibria will not be considered in this paper, the reason being that we want to stick as closely as possible to the general solution procedure outlined in Harsanyi and Selten (1988), a procedure that gives precedence to pure equilibria whenever possible.

Although mixed strategy equilibria will not be considered, we stress that mixed strategies will play an important role in what follows. The reason is that, in the case at hand, a player will typically be uncertain about whether the opponent will commit or not, and such uncertaintly about the opponent's behavior can be expressed by a mixed strategy. Let m_j be a mixed strategy of player j in the game g^2. Because of the linear-quadratic specification of the game, there are only three "characteristics" of m_j that are relevant to player i, viz. w_j the probability that player j waits, μ_j the average quantity to which j commits himself given that he commits himself, and ν_j, the variance of this quantity. Specifically, it easily follows from (2.6)-(2.9) that the expected payoff of player i against a mixed strategy m_j with characteristics (w_j, μ_j, ν_j) is given by

$$u_i(q_i, m_j) = (1 - w_j)(a_i - q_i - \mu_j)q_i + w_j(2a_i - a_j - q_i)q_i/2 \tag{2.10}$$

$$u_i(W_i, m_j) = (1 - w_j)((a_i - \mu_j)^2/4 + \nu_j/4) + w_j(2a_i - a_j)^2/9 \tag{2.11}$$

Note that uncertainty concerning the quantity to which j will commit himself makes it more attractive for player i to wait: ν_j contributes positively to (2.11) and it does not play a role in (2.10). On the other hand, increasing w_j or decreasing μ_j increases the incentive for player i to commit himself.

3 Risk Dominance and the Tracing Procedure

The concept of *risk dominance* captures the intuitive idea that, when players do not know which of two equilibria should be played, they will measure the risk involved in playing each of these equilibria and they will coordinate expectations on the less risky one, i.e. on the risk dominant equilibrium of the pair. The formal definition of risk dominance involves the *bicentric prior* and the *tracing procedure*. The bicentric prior describes the players' initial assessment about the situation. The tracing procedure is a process that, starting from some given prior beliefs of the players, gradually adjusts the players' plans and expectations until they are in equilibrium. It models the thought process of players who, by deductive personal reflection, try to figure out what to play in the situation where the initial uncertainty is represented by the given prior. Below we describe the mechanisms of the tracing procedure as well as how, according to Harsanyi and Selten (1988), the initial prior should be constructed.

First, however, we recall that risk dominance allows a very simple characterization for 2×2 games with two Nash equilibria: the risk dominant equilibrium is that one for which the product of the deviation losses is largest. Consequently, if risk dominance could always be decided on the basis of the reduced game spanned by the two equilibria under consideration (and if the resulting relation would be transitive), then the solution could be found by straightforward computations. Unfortunately, this happy state of affairs does not prevail in general. The two concepts do not always generate the same solution and it is well-known that the Nash product of the deviation losses may be a bad description of the underlying risk situation in general. (See, Carlsson and Van Damme

(1993) for a simple example.) In our companion paper (Van Damme and Hurkens, 1998) we show that also in duopoly games the two concepts may yield different solutions. In the present case, however, the two concepts do generate the same solutions. Since the calculations based on the reduced game are easily performed we do these first.

Consider, first of all, the reduced game spanned by the Cournot equilibrium (q_1^N, q_2^N) and by the Stackelberg equilibrium (q_1^L, W_2) in which firm 1 leads. In this 2×2 game, W_2 weakly dominates q_2^N, hence, the product of the deviation losses associated with the Cournot equilibrium is zero and, in the reduced game, the Stackelberg equilibrium is risk dominant. Exactly the same argument establishes that the Cournot equilibrium is risk dominated by the Stackelberg equilibrium in which firm 2 leads. Next, consider the reduced game where each player is restricted to either committing himself to his Stackelberg quantity or to wait, which is given in Fig. 1[1]

	q_2^L	W_2
q_1^L	D_1, D_2	L_1, F_2
W_1	F_1, L_2	N_1, N_2

Fig. 1: Reduced version of the quantity commitment game.

where L_i, N_i and F_i are as in (2.3) and where D_i denotes player i's payoff in the case of Stackelberg warfare

$$D_i = (a_i - a_j)(2a_i - a_j)/4. \tag{3.1}$$

At the equilibrium where i leads the product of the deviation losses is equal to

$$(L_i - N_i)(F_j - D_j) = a_j^2(2a_i - a_j)^2/1152.$$

[1]This game has also been studied by Dowrick (1986), who concludes "that there is no obvious solution to this game where firms can choose their roles" (p. 259).

Consequently, the product of the deviation losses at (W_1, q_2^L) is larger than the similar product at (q_1^L, W_2) if and only if

$$a_1(2a_2 - a_1) \quad > \quad a_2(2a_1 - a_2),$$

which holds since $a_1 < a_2$. Hence, the product of the deviation losses is largest at the equilibrium where the efficient firm 2 leads: risk considerations based on reduced game analysis unambiguously point into the direction of the Stackelberg equilibrium where the low cost firm leads. As already argued, there is, however, no guarantee that this shortcut indeed identifies the risk dominant equilibrium of the overall game. The only way to find out is by fully solving the entire game. This we do in the next section. In the remainder of this section, we formally define the concepts involved.

Let $g = (S_1, S_2, u_1, u_2)$ be a 2-person game and let m_i be a mixed strategy of player i in g $(i = 1, 2)$. The strategy m_i represents the initial uncertainty of player j about i's behavior. For $t \in [0, 1]$ we define the game $g^{t,m} = (S_1, S_2, u_1^{t,m}, u_2^{t,m})$ in which the payoff functions are given by

$$u_i^{t,m}(s_i, s_j) = (1 - t)u_i(s_i, m_j) + tu_i(s_i, s_j). \tag{3.2}$$

Hence, for $t = 1$, this game $g^{t,m}$ coincides with the original game g, while for $t = 0$ we have a trivial game in which each player's payoff depends only on his own action and his own prior beliefs. Write Γ^m for the graph of the equilibrium correspondence, i.e.

$$\Gamma^m = \{(t, s) : \; t \in [0, 1], s \text{ is an equilibrium of } g^{t,m}\}. \tag{3.3}$$

It can be shown that, if g is a generic *finite* game, then, for almost any prior m, this graph Γ^m contains a unique distinguished curve that connects the unique equilibrium $s^{0,m}$ of $g^{0,m}$ with an equilibrium $s^{1,m}$ of $g^{1,m}$. (See Schanuel et al. (1991) for details.) The equilibrium $s^{1,m}$ is called the *linear trace* of m. If players' initial beliefs are given by m and if players' reasoning process corresponds to that as modelled by the tracing procedure, then players' expectations will converge on the equilibrium $s^{1,m}$ of g.

In this paper we will apply the tracing procedure to the *infinite* game g^2 that was described in the previous section. To our knowledge, ours is the first application of these ideas to a game with a continuum of strategies. For such games, no generalizations of the Schanuel et al. (1991) results have been established yet, but as we will see in the following sections, there indeed exists a unique distinguished curve in the special case analyzed here. Hence, the non-finiteness of the game g^2 will create no special problems.

It remains to specify the players' initial beliefs when they are uncertain about which of two equilibria of g, s or s', should be played. Harsanyi and Selten (1988) argue as follows. Player j, being Bayesian, will assign a subjective probability z_j to i playing s_i and he will assign the complementary probability $z'_j = 1 - z_j$ to i playing s'_i. With these beliefs, player j will play a best response against the strategy $z_j s_i + z'_j s'_i$ that he expects i to play. Assume that j chooses all best responses with equal probability and denote the resulting strategy of j with $b_j(z_j)$. Player i does not know the beliefs z_j of player j and applying the principle of insufficient reason he considers z_j to be uniformly distributed on $[0, 1]$. Writing Z_j for a uniformly distributed random variable on $[0, 1]$, player i will, therefore, believe that he is facing the mixed strategy

$$m_j = b_j(Z_j) \tag{3.4}$$

and this mixed strategy m_j of player j is player i's prior belief about j's behavior in the situation at hand. Similarly, $m_i = b_i(Z_i)$, where Z_1 and Z_2 are independent, is the prior belief of player j, and the mixed strategy pair $m = (m_1, m_2)$ is called the *bicentric prior* associated with the pair (s, s'). Given this bicentric prior m, we say that s *risk* *dominates* s' if $s^{1,m} = s$, where $s^{1,m}$ is the linear trace of m. In case the outcome of the tracing procedure is an equilibrium different from s or s', then neither of the equilibria risk dominates the other. Such a situation will, however, not occur in our 2-stage action commitment game, provided that the costs of the firms are different.

4 Commitment and Risk Dominance

In this section, we prove our main results. Let g_2 be the endogenous commitment game from Section 2. Write S_i for the pure equilibrium in which player i commits to his Stackelberg leader quantity in period 1, $S_i = (q_i^L, W_j)$, and write C for the equilibrium in which each player commits to his Cournot quantity in period 1, $C = (q_1^N, q_2^N)$. We show that both Stackelberg equilibria risk dominate the Cournot equilibrium and that S_2 risk dominates S_1 when $c_2 < c_1$. The first result is quite intuitive: Committing to q_i^N is a weakly dominated strategy and playing a weakly dominated strategy is risky. The proof of this result is correspondingly easy.

Proposition 1 . *In g^2, the Stackelberg equilibrium S_i risk dominates the Cournot equilibrium C $(i = 1, 2)$.*

Proof. Without loss of generality, we just prove that S_1 risk dominates C. We first compute the bicentric prior that is relevant for this risk comparison, starting with the prior beliefs of player 1.

Let player 2 believe that 1 plays $z_2 S_{11} + (1 - z_2) C_1 = z_2 q_1^L + (1 - z_2) q_1^N$. Obviously, if $z_2 \in (0, 1)$, then the best response of player 2 is to wait. Hence, the prior belief of player 1 is that player 2 will wait with probability 1, $m_2 = W_2$.

Next, let player 1 believe that 2 plays $z_1 S_{12} + (1 - z_1) C_2 = z_1 W_2 + (1 - z_1) q_2^N$. Obviously, waiting yields player 1 the Nash payoff N_1 as in (2.3), irrespective of the value of z_1. When $z_1 > 0$ then committing to a quantity that is (slightly) above q_1^N yields a strictly higher payoff, hence, the best response is to commit to a certain quantity $q_1(z_1)$. The reader easily verifies that $q_1(z_1)$ increases with z_1 and that $q_1(1) = q_1^L$. Consequently, if

m_1 is the prior belief of player 2 then for the characteristics (w_1, μ_1, ν_1) of m_1 we have: $w_1 = 0,\ \mu_1 > q_1^N,\ \nu_1 > 0$.

Now, let us turn to the tracing procedure. The starting point corresponds to the best replies against the prior. Obviously, the unique best response against m_2 is for player 1 to commit to q_1^L, while player 2's unique best response against m_1 is to wait. Hence, the unique equilibrium at $t = 0$ is S_1. Since S_1 is an equilibrium of the original game, it is an equilibrium for any $t \in [0, 1]$. Consequently, the distinguished curve in the graph Γ^m is the curve $\{(t, S_1) :\ t \in [0, 1]\}$ and S_1 risk dominates C. $\square$

We now turn to the risk comparison of the two Stackelberg equilibria. Again we start by computing the bicentric prior based on S_1 and S_2. Let player j believe that i commits to q_i^L with probability z and that i waits with probability $1 - z$. From (2.2), (2.10) and (2.11) we obtain

$$
\begin{aligned}
u_j(q_j, zq_i^L + (1-z)W_i) &= z(3a_j - 2a_i - 2q_j/2) + (1-z)(2a_j - a_i - q_j)q_j/2 \qquad (4.1)\\
u_j(W_j, zq_i^L + (1-z)W_i) &= z(3a_j - 2a_i)^2/16 + (1-z)(2a_j - a_i)^2/9 \qquad (4.2)
\end{aligned}
$$

Given z, the optimal commitment quantity $q_j(z)$ of player j is given by

$$q_j(z) = (a_j - a_i)/2 + a_j/2(1+z), \tag{4.3}$$

which results in the optimal commitment payoff equal to

$$[2a_j - a_i + z(a_j - a_i)]^2/8(1+z). \tag{4.4}$$

Note that $q_2(z) > q_1(z)$ for all $z \in [0, 1]$. The reader easily verifies that committing yields a higher payoff than waiting if and only if z is sufficiently small. Specifically, committing is better for player j provided that $z \leq z_j$ where

$$z_j = \frac{(4a_j - 2a_i)^2}{18a_j^2 - (4a_j - 2a_i)^2} \tag{4.5}$$

Note that $0 < z_1 < z_2$, so that both players initially commit with positive probability, it being more likely that player 2 commits. Hence, denoting the best response of player j against $zq_i^L + (1-z)W_i$ by $b_j(z)$, we have

$$b_j(z) = \begin{cases} W_j & \text{if} \quad z > z_j. \\ q_j(z) & \text{if} \quad z < z_j. \end{cases} \tag{4.6}$$

Consequently, writing m_j for the prior of player i (m_j being given by (3.4)) and writing (w_j, μ_j, ν_j) for the characteristics of this prior we have

$$w_j = 1 - z_j, \tag{4.7a}$$

$$\mu_j = (a_j - a_i)/2 + a_j \ln(1 + z_j)/2z_j, \tag{4.7b}$$

$$\nu_j = a_j^2/4(1 + z_j) - a_j^2 \ln^2(1 + z_j)/4z_j^2. \tag{4.7c}$$

Straightforward computations now show that

$$w_1 > w_2, \tag{4.8a}$$

$$\mu_1 < \mu_2, \text{ and} \tag{4.8b}$$

$$\nu_1 < \nu_2. \tag{4.8c}$$

These inequalities already give some intuition for why committing is more risky for player 1: he attaches a smaller probability to the opponent waiting, he expects the opponent to commit to a larger quantity on average, and he is more uncertain about the quantity to which the opponent commits himself. All three aspects contribute positively to making waiting a more attractive strategy.

In the next Lemma we show that actually waiting is a dominant strategy for firm 1 at the start of the tracing procedure whenever the cost differential is sufficiently large. Write

$$\alpha_j = \frac{a_i}{a_j} \tag{4.9}$$

for the relative cost advantage of player i ($\alpha_j > 1$ if and only if $c_i < c_j$). Note that z_j depends on a_i, a_j only through α_j

$$z_j = \frac{(4-2\alpha_j)^2}{18-(4-2\alpha_j)^2} \tag{4.10}$$

and that z_j is a decreasing function of α_j.

Lemma 1 *Write m_2^0 for the prior strategy of player 2 as given by (4.7). If α_2 is sufficiently small, then $u_1(q_1, m_2^0) < u_1(W_1, m_2^0)$ for all q_1. In particular this holds if $z_2 \geq \frac{1}{2}$.*

Proof. We have

$$\begin{aligned} u_i(q_i, m_j^0) &= z_j(a_i - q_i - \mu_j)q_i + (1-z_j)(2a_i - a_j - q_i)q_i/2 \\ &= [a_i - a_j/2 + z_j(a_j/2 - \mu_j)]q_i - (1+z_j)q_i^2/2. \end{aligned} \tag{4.11}$$

Hence, the optimal commitment quantity against the prior is

$$q_i^* = \frac{a_i - a_j/2 + z_j(a_j/2 - \mu_j)}{1+z_j} \tag{4.12}$$

We know that any quantity $q_j \leq q_i^N$ is weakly dominated by W_i for player i in g^2, hence, such a quantity yields strictly less than W_i against any nondegenerate mixed strategy of player j. Consequently, the result follows if $q_1^* \leq q_1^N$. Now, the inequality $q_i^* \leq q_i^N$ is equivalent to

$$2a_i - a_j + z_j(2a_j - a_i) \leq 3a_j \ln(1 + z_j)$$

or

$$2\alpha_j - 1 + z_j(2 - \alpha_j) \leq 3 \ln(1 + z_j) \tag{4.13}$$

A straightforward computation shows that this inequality is satisfied when $z_j = \frac{1}{2}$. (In that case $\alpha_j = 2 - \sqrt{\frac{3}{2}}$.) In the relevant parameter range ($z_j \leq 1, \alpha_j \geq \frac{2}{3}$), the derivative of the LHS of (4.13) (with respect to α_j) is larger than the derivative of the RHS of (4.13), hence, the result follows. □

In the next Lemma we show that, in contrast to the previous result, the most efficient firm's best response to the prior is always to commit.

Lemma 2 *Write m_1^0 for the prior strategy of player 1 as given by (4.7). Then $u_2(m_1^0, W_2) < \max_{q_2} u_2(m_1^0, q_2)$.*

Proof. Substituting (4.12) into (4.11) yields the optimal payoff that player i can get by committing himself

$$u_i^c(m_j^0) = \frac{[a_i - a_j/2 + z_j(a_j/2 - \mu_j)]^2}{2(1 + z_j)}$$

On the other hand, waiting yields

$$u_i(W_i, m_j^0) = z_j[(a_i - \mu_j)^2/4 + \nu_j/4] + (1 - z_j)(2a_i - a_j)^2/9$$

so that by rearranging we obtain

$$u_2^c(m_1^0) - u_2(m_1^0, W_2) = \frac{a_1^2}{4(1+z_1)}\Phi(\alpha_1, z_1) \tag{4.14}$$

where

$$\begin{aligned}\Phi(\alpha, z) \quad = \quad & \frac{1-z}{36}(2 - 8\alpha + 8\alpha^2 + z(-7 + 10\alpha - \alpha^2) + 18(1-\alpha)\ln(z+1)) \\ & -\frac{z}{4} + \frac{\ln^2(1+z)}{2}\end{aligned}$$

and where α_1 and z_1 are as in (4.9) and (4.10). Note that z_1 is a function of α_1, so that Φ (as appearing in (4.14)) can be viewed as a function of α_1 only. A direct computation shows that $\Phi(1) > 0$, hence, player 2 prefers to commit when the costs are equal. In the appendix we show that

$$\Phi_\alpha \geq 0, \Phi_z \leq 0, \text{ and } z_\alpha \leq 0 \tag{4.15}$$

from which it follows that committing becomes more attractive for player 2 when his cost advantage increases. Consequently, firm 2 finds it optimal to commit against the prior for all parameter constellations. □

The Lemmas 1 and 2 imply that the Stackelberg equilibrium with firm 2 as leader is the (unique) equilibrium at the start of the tracing procedure when $z_2 \geq \frac{1}{2}$. It, hence, is an equilibrium of g^t for any value of t and, therefore

Corollary 1 *If the difference in costs is sufficiently large (specifically, if $z_2 \geq \frac{1}{2}$), then the Stackelberg equilibrium in which the efficient firm leads risk dominates the other Stackelberg equilibrium.*

In the remainder of this section, we will confine attention to the case where the cost difference is small enough so that also for the inefficient firm 1 the best response to the prior involves a commitment. So from now on $z_2 < \frac{1}{2}$.[2] The next Lemma shows that it cannot be true that both firms keep on committing themselves to the end of the tracing procedure: at least one of the firms has to switch. The Lemma thereafter will then show that it is the weakest firm that switches first, which implies that the outcome will always be leadership of the strong firm.

Lemma 3 *Let s^t be the equilibrium on the path of the tracing procedure at "time" t if the players priors are as in (4.7). Then there exists $i \in \{1, 2\}$ and $t < 1$ such that $s_i^t = W_i$.*

Proof. Assume not, so that each player finds it optimal to commit at each point reached by the tracing path. Writing q_i^t for the optimal commitment quantity of player i at time t, it is easily seen that $q_i^1 = q_i^N$ for $i = 1, 2$, since the payoff functions at $t = 1$ coincide with those of the original game. Furthermore, $q_i^t > q_i^N$ for $t < 1$ since any quantity less than a_i^N is strictly dominated by waiting. Write u_i^t for the payoff function at "time" t when the prior is given by (4.7) and let

$$g_i(t) = u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t) \tag{4.16}$$

be the gain that player i realizes by committing himself. Clearly, $g_i(1) = 0$. Furthermore, by the envelope theorem

[2]This bound is not sharp. It can be shown that committing is optimal for firm 1 if $\alpha_1 > 1.081$ (or $z_2 < \frac{7}{20}$).

$$g_i'(t) = \frac{\partial}{\partial t}\left[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)\right] + \frac{\partial}{\partial q_j}\left[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)\right]\frac{\partial q_j^t}{\partial t}.$$

For $t = 1$, the first term in this expression is equal to

$$u_i(q_i^N, q_j^N) - u_i(W_i, q_j^N) - u_i(q_i^N, m_j^0) + u_i(W_i, m_j^0) = u_i(W_i, m_j^0) - u_i(q_i^N, m_j^0) > 0.$$

Furthermore, the partial derivative with respect to q_j is equal to

$$-q_i^N + (a_i - q_j^N)/2 = 0$$

so that $g_i'(1) > 0$ and $g_i(t) < 0$ for some $t < 1$. But this contradicts our assumption that it is optimal to commit for each player for any value of $t < 1$. □

Our strategy for proving that it is the weakest firm that switches first is to show that this firm will switch first even when the more efficient firm is more 'pessimistic'. Specifically, we will show that even when the efficient firm believes that the other commits with the same probability as it itself does, the inefficient firm will switch before. Specifically, write m_j for the prior strategy of player j as given by (4.6) and write $\bar{m}_j$ for the strategy defined similarly, but with z_1 replaced by z_2. Let $m = (m_1, m_2)$ and $\bar{m} = (\bar{m}_1, \bar{m}_2)$. Hence, player 2 is more pessimistic in $\bar{m}$, while player 1's prior beliefs are the same in m and $\bar{m}$. (Recall from (4.5) that $z_1 < z_2$.) Assume that each player finds it optimal to commit at $t = 0$ when the prior is m. Write $q_i^{t,m}(q_j)$ for the best commitment quantity of player i at t when the opponent commits to q_j at that time and denote the (unique) pair of mutual best commitment quantities by (q_1^t, q_2^t). Write

$$g_i^t(q_i, q_j) = u_i^{t,m}(q_i, q_j) - u_i^{t,m}(W_i, q_j). \tag{4.17}$$

then $g_i^t(q_i^t, q_j^t) > 0$ $(1 = 1, 2)$ for t sufficiently small and (q_1^t, q_2^t) is the equilibrium on the tracing path for such t. Define $(\bar{q}_1^t, \bar{q}_2^t)$ and $\bar{g}_i^t$ similarly, but with m replaced by $\bar{m}$ in the above definitions. We now have

Lemma 4 *Let* $t_i = \sup\{\tau \in [0,1] : \ g_i^t(q_i^t, q_j^t) \geq 0$ *for all* $t \in [0,\tau]\}$. *Then* $t_2 > t_1$.

Proof. We only provide a sketch of the proof here and relegate technical details to the Appendix. The proof consists of comparing the tracing path (q_1^t, q_2^t) with the tracing path $(\bar{q}_1^t, \bar{q}_2^t)$. We first show that $\bar{q}_2^t < q_2^t$ and $\bar{q}_1^t \geq q_1^t$. These inequalities are intuitive: player 2 is more pessimistic if the prior is $\bar{m}$, hence, he will commit to a lower quantity. This in turn gives player 2 an incentive to commit to a higher quantity when the prior is $\bar{m}$. Furthermore, if player 2 is more pessimistic, then he finds committing himself less attractive: $\bar{g}_2^t \leq g_2^t$. Still, since firm 2 has lower cost than firm 1 has, committing is more attractive for firm 2 than for firm 1 when both firms are equally pessimistic: $\bar{g}_1^t < \bar{g}_2^t$. The result follows by combining the above observations. Formally then, in the Appendix we establish the following inequalities:

$$\begin{aligned} g_1^t(q_1^t, q_2^t) = \bar{g}_1^t(q_1^t, q_2^t) &\overset{(1)}{\leq} \bar{g}_1^t(q_1^t, \bar{q}_2^t) \overset{(2)}{\leq} \bar{g}_1^t(\bar{q}_1^t, \bar{q}_2^t) \overset{(3)}{<} \\ \bar{g}_2^t(\bar{q}_1^t, \bar{q}_2^t) &\overset{(4)}{\leq} \bar{g}_2^t(q_1^t, \bar{q}_2^t) \overset{(5)}{\leq} g_2^t(q_1^t, \bar{q}_2^t) \overset{(6)}{\leq} g_2^t(q_1^t, q_2^t). \end{aligned} \tag{4.18}$$

(The first equality holds since player i's prior is the same in both cases; the first and fourth inequality follow from the monotonicity of the quantities; the second and sixth inequality follow from the best response properties, and the fifth inequality follows since player 2 is more pessimistic when the prior is $\bar{m}$.) $\square$

Lemma 4 implies that at $t = t_1$ the tracing path reaches the equilibrium $(q_1^{t_1}, q_2^{t_1})$, with player 1 being actually indifferent between waiting and committing to $q_1^{t_1}$. The tracing path must now continue along an interval I (with $t_1 \in I$) with equilibria of the form $(m_1(t), q_2^I(t))$, where player 2 commits to $q_2^I(t)$ and player 1 uses a mixed strategy: he waits with probability $w(t)$ and commits to $q_1^I(t)$ with the complementary probability

$1-w(t)$. The two commitment quantities are determined by the optimality condition for player 1 ($q_1^I(t)$ must be the optimal commitment quantity) and the indifference condition for player 1 (committing optimally yields the same payoff as waiting). The probability of waiting, $w(t)$, is determined by the optimality condition for player 2.

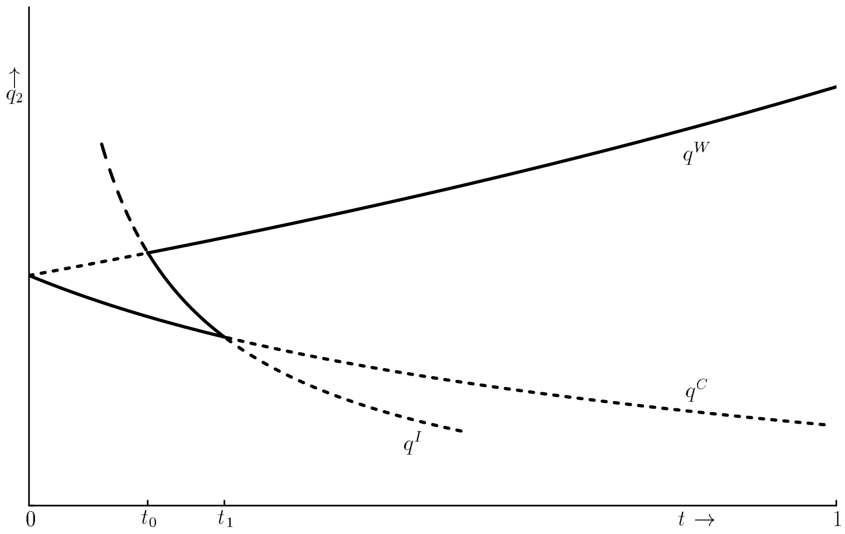

Fig. 2: The tracing path initially follows q^C, then bends backwards along q^I and finally ends along q^W at S_2.

Fig. 2 illustrates the argumentation: Time t is on the horizontal axis, firm 2's commitment quantity on the vertical axis. The Figure contains three curves. Curve q^C plots the commitment strategy of firm 2 when firm 1 commits for sure and play is in equilibrium. As we established in Lemma 4, both firms keep committing from $t = 0$ to $t = t_1$, therefore the tracing path follows this curve upto $t = t_1$. Curve q^I plots firm 2's commitment quantity that leaves firm 1 exactly indifferent between committing and waiting. The tracing path has to continue along this curve from $t = t_1$. (In the Appendix we establish that the curve necessarily bends backwards.) Curve q^W describes the optimal commitment quantity when firm 1 waits with probability 1. The tracing path follows this curve from $t = t_0$ to $t = 1$. It follows that the endpoint of the tracing path is the equilibrium where player 2 leads, hence, we have shown

Proposition 2 *The Stackelberg equilibrium in which the low cost firm leads risk dominates the Stackelberg equilibrium in which the efficient firm follows.*

By combining the Propositions 1 and 2 we, therefore, obtain our main result:

Theorem 1 *The Stackelberg equilibrium in which the efficient firm leads and the inefficient firm follows is the risk dominant equilibrium of the endogenous quantity commitment game.*

Furthermore, as a Corollary we immediately have that the shortcut via the reduced games, as taken in Section 3, indeed correctly identified the risk dominant equilibrium of the overall game. Finally, the Stackelberg equilibrium that is selected is the one with the highest produced quantity (hence, the lowest price) and the highest total profits. So, in this case, the selected equilibrium is the one where both the producer and the consumer surplus are highest.

5 Conclusion

In this paper we have endogenized the timing of the moves in the linear quantity-setting duopoly game by means of Harsanyi and Selten's concept of risk-dominance. To our knowledge, this is the first application of the (linear) tracing procedure to games where the strategy spaces are not finite[3]. We have seen that no new conceptual problems are encountered, but that the computational complexities are quite demanding. Ex post we could verify that these computations were not necessary: The shortcut by means

[3]Harsanyi and Selten (1988) and Güth and Van Damme (1991) considered discretized versions of games with infinite strategy sets.

of a comparison of the Nash products of the deviation losses yields the same answer. However, as already said, there is no guarantee for this to happen in general and in our companion paper Van Damme and Hurkens (1998) we show that the two concepts yield different solutions in a price setting context. In that paper we analyze endogenous price leadership in a linear market for differentiated products. Again, we assume that firms differ in their marginal costs and we show that the efficient firm is the leader in the risk dominant equilibrium. In this case, however, that equilibrium has a smaller Nash product than the Stackelberg equilibrium in which the inefficient firm leads. Quite interestingly, if the cost differential is sufficiently small, the inefficient firm has higher profits than the efficient firm in the risk dominant equilibrium: It profits from free riding as a follower.

Although Von Stackelberg (1934) argued that in general it is not possible to determine theoretically which of the duopolists will become the leader ("Es is jedoch theoretisch nicht zu entscheiden, welcher der beiden Dyopolisten obsiegen wird", p. 20), he also provides a numerical example for which he does determine the actual leader. The example is given by

$$p = 10 - Q/100, \;\; c_1 = 2, \;\; c_2 = 1.5, \;\; F_1 = 500, \;\; F_2 = 600, \tag{5.1}$$

where F_i is the (unavoidable) fixed costs of firm i. Von Stackelberg argues that in this case firm 2 (which is the one with the lower marginal cost) will most likely become the market leader since it makes less losses than firm 1 in the case of Stackelberg warfare: We have $q_1^L = 375, q_2^L = 450, D_1 = -593.75, D_2 = -487.50$. Hence, firm 2 makes less losses during the price war and, therefore, it can win the war of attrition. Of course, this argument is entirely different from the one developed in this paper. Von Stackelberg also remarks that actually this outcome is quite natural and follows from the model's assumption that the second firm is a more modern one which has higher fixed costs, but lower marginal cost.[4] This last comment is very intriguing since, if the modern firm

[4]Von Stackelberg denotes the first firm by A and the second by B and he writes "In unserem Beispiel

would have substantially higher fixed costs, exactly the same argument would imply that the old-fashioned firm would become the leader.

Note that we did not provide the solution of the endogenous timing game for the case where both firms have the same marginal cost. The reader might conjecture that in that case the Cournot equilibrium would be selected, however, Lemma 3 shows that that conjecture is wrong. If the outcome of the tracing procedure at $t = 1$ would be (q_1^N, q_2^N), then each player would strictly prefer to wait at $t < 1$, but clearly (W_1, W_2) cannot be an equilibrium at such t. It follows that, in the symmetric case, the outcome must be a mixed strategy equilibrium. (It obviously must be a symmetric equilibrium as well.) Since mixed equilibria have received almost no attention in the oligopoly literature, we refrain from providing the explicit solution of the symmetric game. Let us note, however, that also in the case where the costs differ, the endogenous timing game has a variety of mixed strategy equilibria. We did not take these into consideration since the Harsanyi and Selten (1988) equilibrium selection theory allows us to neglect them. That theory gives precedence to pure equilibria whenever these exist and we did consider all pure equilibria in this paper.

In this paper we only allowed for one point in time where the players can commit themselves, however, one can easily define the game g^t in which there are $(t-1)$-periods in which the players can commit themselves. ($g^1 = g$, g^2 is as in (2.6) – (2.9) and g^t is defined by induction for $t \geq 3$.) Knowing the solution of g^2, the game g^t, with $t \geq 3$, can be solved by backward induction, i.e. by applying the subgame consistency principle from Harsanyi and Selten (1988): No matter what the history has been, a subgame g^τ has to be played according to its solution. Adopting this principle, one sees that in g^3 waiting is a dominant strategy of player 2: If he waits he can best respond if the opponent commits, while he is guaranteed his Stackelberg leader payoff if the other waits

wird warscheinlich die Unternehmung A der Underlegene sein, weil sie den größeren Verlust erleidet. Dies entspricht auch der Konstruktion unseres Beispiels, in welchem für B ein modernerer Betrieb (höhere fixe Kosten, dafür niedriger proportionaler Satz) angenommen wurde." (Von Stackelberg, 1934, p. 66).

as well. Consequently, player 2 will wait and committing becomes a riskless strategy for player 1. Hence, the solution of g^3 is that player 1 will commit itself. In other words, player 1 commits in order to prevent that player 2 will commit himself. We come to the conclusion that the predicted outcome is very sensitive to the number of commitment periods: If t is even, the solution of g^t is (W_1, q_2^L) while, if $t \geq 3$ is odd, the solution of g^t is (q_1^L, W_2). In our opinion, this lack of robustness reflects the fact that the discrete time model with $t \geq 3$ is not an appropriate one to model commitment possibilities. In future work we plan to investigate the issue in continuous time, while possibly also allowing for commitments to be built up gradually. For earlier work along this direction, we refer to Spence (1972) and to Fudenberg and Tirole (1983).

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Appendix

In this Appendix we complete the proofs of the Lemmas 2 and 4 and Proposition 2.

PROOF OF LEMMA 2

We have to show that the inequalities (4.16) hold. Hence, we have to show that

$$\Phi_\alpha \geq 0,\ \ \Phi_z \leq 0,\ \ z_\alpha \leq 0,$$

where

$$\begin{aligned}\Phi(\alpha,z) \;\; = \;\; & \frac{1-z}{36}\Big(2-8\alpha+8\alpha^2+z(-7+10\alpha-\alpha^2)+18(1-\alpha)\ln(z+1)\Big) \\ & -\frac{z}{4}+\frac{\ln^2(z+1)}{2}.\end{aligned}$$

It is straightforward to verify that $z_\alpha \leq 0$. It is easily seen that $\Phi_\alpha \geq 0$ if and only if

$$-8+16\alpha+z(10-2\alpha)-18\ln(z+1)\geq 0.$$

Now, $\ln(1+z)\leq z$, $z\leq {}^1\!/_2$, and $\alpha\geq 1$, from which it follows that the above inequality holds for all (α, z)-combinations in the relevant domain. Next, we have that

$$\begin{aligned}\Phi_z \;\; = \;\; & \frac{-1}{36}\Big(2-8\alpha+8\alpha^2+z(-7+10\alpha-\alpha^2)+18(1-\alpha)\ln(z+1)\Big) \\ & -\frac{1}{4}+\frac{\ln(z+1)}{z+1}+\frac{1-z}{36}\Big(-7+10\alpha-\alpha^2+18(1-\alpha)/(z+1)\Big) \\ \Phi_{z\alpha} \;\; = \;\; & \frac{-1}{36}\left(-8+16\alpha+z(10-2\alpha)-18\ln(z+1)\right)+\frac{1-z}{36}\left(10-2\alpha-\frac{18}{z+1}\right)<0.\end{aligned}$$

Hence,

$$\Phi_z(\alpha,z)\leq\Phi_z(1,z)=-\frac{1}{4}-\frac{z}{9}+\frac{\ln(z+1)}{z+1}<0$$

where the last inequality follows from $z \leq {}^1\!/_2$.

This completes the proof of the inequalities (4.16) and, therefore, of Lemma 2. □

PROOF OF LEMMA 4

We will prove the inequalities from (4.19) in the following order: First (1) and (4), next (5) and finally (3). Note that the inequalities (2) and (6) hold by definition: $\bar{q}_1^t$ (resp. q_2^t) is the optimal commitment quantity against $\bar{q}_2^t$ (resp. q_1^t) in the game $\bar{g}^t$ (resp. g^t). Furthermore, the equality in (4.16) holds since player 1 has the same prior in g as in $\bar{g}$.

PROOF OF THE INEQUALITIES (1) AND (4) FROM (4.19)

Write m_j^x for the mixed strategy of player j defined by

$$m_j^x = \begin{cases} q_j(z) \text{ as in (4.3)}, & \text{if } z \leq x, \\ W_j & \text{otherwise.} \end{cases}$$

Hence, $m_j^{z_j} = m_j$ and $m_j^{z_i} = \bar{m}_j$. Write $g^{t,x}$ for the game at t when the prior is given by m^x. It is easily verified that player i's optimal commitment quantity in $g^{t,x}$ against q_j is given by

$$q_i^{t,x}(q_j) = \frac{(1-t)[a_i - a_j/2 + x(a_j/2 - \mu_j^x)] + t[a_i - q_j]}{(1-t)(1+x) + 2t},$$

where μ_j^x is as in (4.7b) but with z_j replaced by x. Substituting that expression for μ_j^x and rewriting yields

$$q_i^{t,x}(q_j) = \frac{a_i + (1-t)[xa_i - a_j - a_j \ln(1+x)]/2 - tq_j}{1 + t + x(1-t)}$$

Claim 1

$$\frac{\partial q_2^{t,x}(q_1)}{\partial x} < 0 \text{ if } x \leq {}^1\!/_2,\ q_1 \leq q_1^L \text{ and } t < 1$$

.

A straightforward computation shows that the derivative is negative if and only if $t < 1$ and

$$\left(a_2 - \frac{a_1}{1+x}\right)(1 + t + x(1-t)) < 2a_2 + (1-t)(xa_2 - a_1 - a_1 \ln(1+x)) - 2tq_1,$$

and, as both sides of this inequality are linear in t, it suffices to check that the inequality holds at both endpoints. Now, at $t = 0$ the inequality simplifies to

$$a_1 \ln(1+x) < a_2$$

which holds since $a_1 \leq a_2$. At $t = 1$, the inequality simplifies to

$$q_1 < a_1(1+x)$$

and this holds because of our restrictions on the parameters. (Recall that these restrictions are without loss of generality: player 1 will not commit to a quantity that is larger than the Stackelberg leader quantity and if $z_2 \geq 1/2$ then Lemma 1 applies.)

Since $z_1 < z_2$ (cf. (4.8)), Claim 1 implies that player 2's best response quantity is lower in $g^{t,\bar{m}}$ than it is in $g^{t,m}$. Since player 1 has the same best response correspondence in these two games, it follows that player 2 (resp. player 1) commits to a lower (resp. higher) quantity in $g^{t,\bar{m}}$ than in $g^{t,m}$. (Formally, if $t \leq \min(t_1, t_2)$, then the map $q_2 \to q_2^{t,m} q_1^{t,m}(q_2)$ is increasing and cuts the 45°-degree line at a point lower than the one where the first graph cuts the diagonal.) Hence, Claim 1 establishes that for $t < 1$:

$$q_1^t < \bar{q}_1^t \text{ and } q_2^t > \bar{q}_2^t.$$

The proof of the inequalities (1) and (4) can now be completed by showing that the gain from committing is decreasing in the opponent's quantity. Because of the linearity of the payoff function in t it suffices to show that this holds for $t = 1$, i.e. for the original game. Now

$$\begin{aligned}\frac{\partial}{\partial q_j}u_i(q_i,q_j) &= -q_i,\\ \frac{\partial}{\partial q_j}u_i(W_i,q_j) &= -(a_i-q_j)/2.\end{aligned}$$

In the relevant range where both players find it optimal to commit themselves ($t \leq \min(t_1, t_2)$) we have $q_i \geq q_i^N$ for $i = 1, 2$, and, therefore

$$-q_i + (a_i - q_j)/2 \leq 0,$$

which completes the inequalities (1) and (4).

PROOF OF INEQUALITY (5)

We have that

$$\begin{aligned} & (1-t)^{-1}[g_2^t(q_1,q_2) - \bar{g}_2^t(q_1,q_2)] = \\ = & \int_{z_1}^{z_2}[u_2(W_1,q_2) - u_2(q_1(z),q_2)]dz + \int_{z_1}^{z_2}[u_2(q_1(z),W_2) - u_2(W_1,W_2)]dz \\ = & \int_{z_1}^{z_2}[u_2(W_1,q_2) - u_2(W_1,W_2)]dz + \int_{z_1}^{z_2}[u_2(q_1(z),W_2) - u_2(q_1(z),q_2)]dz \end{aligned}$$

The second integrand is clearly nonnegative. The first is nonnegative since $q_2 \geq q_2^N$. This establishes inequality (5).

PROOF OF INEQUALITY (3)

The proof involves some straightforward, but tedious calculations. For simplicity, write $x = z_2$. Because of Lemma 1 and Corollary 1 we may confine ourselves to the case where $x < 1/2$. The reader may verify, that up to a positive multiplier, $\bar{g}_2^t(\bar{q}_1^t, \bar{q}_2^t) - \bar{g}_1^t(\bar{q}_1^t, \bar{q}_2^t)$ is equal to $\Psi(t, x)$, where

$$
\begin{aligned}
\Psi(t,x) &= (x+1)(x-2)(-2x-1) + t(x-1)^2(4x+1) \\
&\quad + t^2(-3 - x - x^2 + 4x^3 - 2x^4)/(1+x) \\
&\quad + 6(x^2 - 1 + t(-1 + 2x - 2x^2) + t^2(x-1)^2 \ln(1+x) \\
&\quad + 3(t-1)(2 + 2x + t(3-2x)) \ln^2(1+x)
\end{aligned}
$$

For $x \leq 1/2$, $\Psi(t,x)$ is concave in t so that the minimum is attained in $t = 0$ or $t = 1$. Now direct substitution yields

$$
\Psi(0,x) = (x+1)(x-2)(-2x-1) + 6(x^2-1)\ln(1+x) - 6(1+x)\ln^2(1+x)
$$

Using the fact that $\ln(1+x) \leq x$, we obtain

$$
\begin{aligned}
\Psi(0,x) &\geq (x+1)(x-2)(-2x-1) - 6(1-x^2)x - 6(1+x)x^2 \\
&= (x+1)(2 - 3x - 2x^2) > 0
\end{aligned}
$$

Another direct substitution gives

$$
\begin{aligned}
(1+x)\Psi(1,x) &= 9x - 6(1+x)\ln(1+x) \\
&\geq 3x(1-2x) > 0,
\end{aligned}
$$

where we again have used that $\ln(1+x) \leq x$. Consequently $\Psi(t,x) > 0$ for all t and x, which completes the proof of inequality (3). $\square$

PROOF OF PROPOSITION 2

We first recall that in g^{t,m^0} the optimal commitment quantity against q_j is

$$
q_i^{t,m^0}(q_j) = \frac{(1-t)(a_i - a_j/2 + z_j(a_j/2 - \mu_j)) + t(a_i - q_j)}{1 + t + (1-t)z_j} \tag{A.1}
$$

Let $q_1^C(t)$ and $q_2^C(t)$ be optimal commitment quantities against each other. Using (A.1) (applied to $q_i^C(t)$) we can rewrite it as

$$q_i^{t,m^0}(q_j) = \frac{-t(a_i - q_j^C(t)) + q_i^C(t)(1 + t + (1-t)z_i) + t(a_i - q_j)}{1 + t + (1-t)z_j} \tag{A.2}$$

For $t \in (0,1)$, let $q_2^I(t)$ denote the commitment quantity of firm 2 that leaves firm 1 indifferent between committing optimally (to $q_1^I(t)$) and waiting. We know from previous analysis that firm 1 strictly prefers committing to waiting when firm 2 commits to $q_2^C(t)$, for all $t < t_1$. Moreover, the gain from committing is decreasing in the opponent's commitment strategy. Hence, the curve $q_2^I(t)$ intersects the curve $q_2^C(t)$ from above at $t = t_1$. (See Fig. 2.)

The tracing path must continue along the curve $q_2^I(t)$ for some time. We need to establish the direction. On the tracing path it must hold that $q_2^I(t)$ is the best reply against firm 1's strategy of waiting with probability $w(t) \geq 0$ and committing with the remaining probability to $q_1^I(t)$. It is easily established that the optimal commitment strategy of firm 2 is increasing in $w(t)$ (keeping firm 1's quantity fixed). So $q_2^I(t) \geq q_2^{t,m^0}(q_1^I(t))$. Using (A.2) this is equivalent to

$$(1 + t + (1-t)z_1)q_2^I(t) \geq -t(a_2 - q_1^C(t)) + q_2^C(t)(1 + t + (1-t)z_2) + t(a_2 - q_1^I(t)).$$

Multiplying both sides by $1 + t + (1-t)z_2$ and using (A.2) once more, this is equivalent to

$$\begin{aligned}&(1 + t + (1-t)z_2)(1 + t + (1-t)z_1)q_2^I(t) \geq \\ &\quad q_1^C(t)t(1-t)(z_2 - z_1) + q_2^C(t)((1 + t + (1-t)z_2)^2 - t^2) + t^2 q_2^I(t),\end{aligned}$$

which, since $z_2 > z_1$, implies that

$$q_2^I(t) \geq q_2^C(t).$$

This implies that the tracing path must bend backwards. $\square$