Commitment Robust Equilibria and Endogenous Timing*

Eric van Damme[†] Sjaak Hurkens[†]

First version July 1993
revised July 1994

Abstract

This paper investigates which equilibria of a game are still viable when players have the opportunity to commit themselves. To that end we study a model of endogenous timing in which players face the trade-off between committing early and forcing the opponent to best respond, and moving late so as to be able to play a best response against the opponent. It is shown that most mixed strategy equilibria of the original game are not viable when the sequencing of the moves is endogenous. In contrast, any pure strategy equilibrium is a perfect equilibrium outcome of the timing game. More refined equilibrium concepts with an evolutionary flavor, however, allow the conclusion that only equilibria in which no player has an incentive to move first are viable.

This paper is a substantial revision of our paper "Endogenous Timing and Strategic Commitment" that was circulated in October 1992.

[†]CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. The research of Sjaak Hurkens was sponsored by the Foundation for the Promotion of Research in Economic Sciences, which is part of the Netherlands Organization for Scientific Research (NWO). We thank Werner Güth and two anonymous referees for useful comments.

1 Introduction

One of the most important ideas in game theory is the value of commitment, the idea that it may be advantageous to constrain one's own behavior in order to induce others to behave in a way that is favorable to oneself. Schelling's (1960) classic *The Strategy of Conflict* is filled with examples that illustrate this idea that it might pay to reduce one's flexibility, that it may be optimal to burn the bridges behind oneself. The simplest commitment possibility that Schelling discusses (and what he calls the "pure unconditional commitment") is equivalent to obtaining the first move: to preempt one's rivals by choosing and communicating an action before they do. In the economics literature, this first-mover advantage has been known at least since Von Stackelberg (1934) pointed out that in a quantity setting duopoly game, the leader has higher profits than the follower and than a Cournot competitor. Various authors have argued that the fact that each duopolist has an incentive to move earlier than his opponent makes the Cournot equilibrium somewhat suspect. Of course, if the duopolists are indeed forced to move simultaneously (as the standard game model of the duopoly situation assumes), then there is nothing wrong with the Cournot equilibrium, but one may wonder whether in real situations the rules are indeed that rigid as to prevent commitments from being made.

The above observation naturally leads to the more general question of which Nash equilibria are still viable when players have the opportunity to commit themselves. This question has been addressed recently in Rosenthal (1991). Rosenthal defines an equilibrium s of a 2-person game g to be commitment robust if s is also a subgame perfect equilibrium outcome of each of the two games where one of the players moves first. He argues that "failure of commitment robustness ought to signal that there is a sense in which the equilibrium is questionable if in the intended application the rules can be modified in the appropriate way sufficiently inexpensively by the player who stands to gain". However, this argument is not entirely compelling. Rosenthal compares the simultaneous move game with the two perfect information sequential move games in which the leader is exogenously specified, but it is not clear why the latter games are relevant to the study of the former situation. The original problem derives from the possibility that each player might or might not choose to move before the other. Both players simultaneously have the possibility to commit themselves, no player can unilaterally change the rules of the game, hence, one would like to see the commitments arise endogenously. Rosenthal seems well aware of this problem with the definition that he proposes. He writes: "In defining commitment-robustness, one might require consideration of more than just the alternative games G_I and G_{II}; after all there could be opportunities for both players to invest in commitment possibilities. It seems best, therefore, to think of the defining conditions here as being in the nature of necessary conditions."

The question we would like to raise is whether Rosenthal's definition indeed gives necessary conditions. Specifically, could it not be that, even though each player could profit from moving first, no player dares to commit since he fears that the opponent might commit simultaneously? Hence, could it not be that, as a consequence, the players end up in the Nash equilibrium after all? Certainly, the latter seems a real possibility in cases where committing to an action is risky, as it is, for example, in the quantity setting duopoly game. To address these issues we make use of the game of action commitment that was proposed in Hamilton and Slutsky (1990). The rules are as follows: There are two periods and each player has to move in exactly one of these periods. Choices are simultaneous, but, if one player chooses to move early (i.e. to commit) while the other moves late, the latter is informed about the former's choice before making his decision. Hence, in this game, each player can commit, no player has the sole privilege of being able to do so. We investigate which equilibria of the original game can arise as "sensible" equilibrium outcomes of the action commitment game. Are indeed only those equilibria viable at which no player has an incentive to move first? Is commitment robustness, as defined by Rosenthal, a necessary condition for equilibrium outcomes to survive when the sequencing of the moves is endogenous?

A first important result is that a mixed strategy equilibrium indeed is viable only if no player has an incentive to move first at this equilibrium (Proposition 2). Only in this case does the action commitment game have a subgame perfect equilibrium that produces this outcome. Hence, endogenous timing will eliminate most mixed strategy equilibria. The intuition is quite straightforward: The actions actually resulting from the players' mixed strategies need not be in equilibrium, ex post players have an incentive to deviate. Hence, each player will have an incentive to wait, thereby guaranteeing that he is best responding no matter which action the opponent is actually choosing. But if a player waits, the opponent frequently will prefer to commit to his Stackelberg leader strategy.

For pure strategy equilibria, the situation is much different and it turns out that these are not as easily eliminated: An immediate commitment of each player to this equilibrium is part of a subgame perfect equilibrium of the 2-stage action commitment game (Proposition 3). Furthermore, such an immediate commitment constitutes a perfect equilibrium in the normal form of the action commitment game (Proposition 4). Intuitively, if each player expects an unattractive outcome in case the timing game reaches the second stage, then it is optimal for each player to commit to the pure equilibrium immediately if he expects his opponent to do the same. Hence, requiring perfectness does not allow one to conclude that only pure equilibria in which no player has an incentive to move first are viable when the timing is endogenous. Consequently Rosenthal's conclusion appears premature.

To see whether Rosenthal's intuition can be given some foundation we then turn to more refined equilibrium notions. The literature offers two types of refinements, the distinction being whether an evolutionary or an eductive interpretation of equilibria is adopted. An example that we discuss extensively in the next section shows that the eductive perspective does not allow one to conclude that only commitment robust equilibria are viable when the timing of the moves is endogenous: Also other pure equilibria may result from proper (Myerson (1978)) and even stable (Kohlberg and Mertens (1986)) equilibria of the action commitment game.

The main result of this paper, however, shows that Rosenthal's intuition can be supported by a variety of set-valued solution concepts that have an evolutionary flavor. These concepts seem to correspond more closely to the interpretation of an equilibrium as a fixed point of an unspecified dynamic process, than to the interpretation of an equilibrium as a self-enforcing agreement. Specifically, our main result shows that the concepts of persistent equilibria (Kalai and Samet (1984)) and curb*-equilibria (Basu and Weibull (1991)) force players to coordinate on the commitment robust equilibrium whenever this is unique and pure (Proposition 7). Hence, if we accept curb* or persistent equilibria as the relevant solution concept, the results in this paper allow us to identify two classes of games for which we can unambiguously determine the outcome if the order of the moves is endogenous, viz. zero-sum games and games with common interest. While the result concerning zero-sum games is not surprising, it is remarkable that one has to turn to very restrictive equilibrium notions to "justify" playing the Pareto-efficient equilibrium in common interest games.

The remainder of the paper is organized as follows. In Section 2 we give an example of a coordination game to show that the intuition, that only commitment robust equilibria are viable when the timing is endogenous, might be misleading. In Section 3 we introduce notation and give a definition of commitment robust equilibria that differs slightly from that of Rosenthal. We compare the two concepts and derive characterizations of them. Section 3 also formally introduces the 2-stage action commitment game. In Section 4 we show that a mixed strategy equilibrium is viable with endogenous timing only if it yields each player at least his Stackelberg leader payoff. In contrast it is shown in Section 5 that requiring perfectness in the timing game does not eliminate pure equilibria of the underlying game. In this section we also discuss the related work of Hamilton and Slutsky (1990). Section 6 investigates whether players will automatically coordinate on a commitment robust equilibrium when the order of the moves is endogenous. Section 7 concludes and discusses some related work.

2 A coordination game

In this section we discuss a simple example to show that the intuition that, when the timing of the moves is endogenous, only equilibria in which no player has an incentive to move first are viable, is not completely reliable. We show that one needs to employ quite restrictive solution concepts in order to arrive at this conclusion. Specifically, the following example demonstrates that even if a game has a strict equilibrium that is also the unique Stackelberg equilibrium outcome in each of the two games where one of the players is forced to move first, it is by no means obvious that players will necessarily coordinate on this equilibrium when the order of the moves is determined endogenously.

Consider the common interest coordination game g given in Table 1a. When players have to move simultaneously, there are three equilibria: (T, L), (B, R) and a mixed equilibrium that yields each player the payoff $\frac{2}{3}$. When one of the players is forced to move first, with this player's choice being revealed to the other, the subgame perfect outcome is (T, L), irrespective of which player moves first. According to Rosenthal's definition only the equilibrium (T, L) is commitment robust.

	L		R	
T	2	2	0	0
B	0	0	1	1

Table 1a: A coordination game.

	L^1		L^2		R^2		R^1	
T^1	2	2	2	2	2	2	0	0
T^2	2	2	2	2	0	0	1	1
B^2	2	2	0	0	1	1	1	1
B^1	0	0	1	1	1	1	1	1

Table 1b: The (reduced) normal form of the 2-period game of endogenous timing associated with the game from Table 1a.

Let us investigate whether this equilibrium is the unique "sensible" outcome of the 2-stage game of action commitment, described in the introduction, in which the order of the moves is endogenously determined. In this action-commitment game both players have 10 pure strategies. (A player has two possibilities to commit himself; if he does not commit he might be in three different information sets in period 2 and in each of these he can choose between two possible actions.) Table 1b displays a reduced form of this normal form: It only lists those strategies in which a player plays a best response in the second period whenever the opponent has unilaterally committed himself in the first period. Since this reduction makes it easier to see the intuition for our results, our discussion in this section will be based on it. Note, however, that our formal results are based on the full, unreduced, game. In Table 1b, X^1 denotes the pure strategy "commit to X in period 1", while X^2 denotes "wait till period 2, play the unique best response if the opponent has already moved in period 1 and, otherwise, play X." The question we want to address is whether "2,2" (i.e. the commitment robust equilibrium) is the unique "sensible" outcome in the game of Table 1b? Alternatively, can one perhaps also give good arguments in favor of "1,1", or in favor of the mixed equilibrium?

Inspection shows that the game from Table 1b has three Nash equilibrium outcomes: There is a connected set of equilibria with payoff (2,2), there is also a connected set of equilibria with payoff (1,1) and there is a completely mixed equilibrium in which each player plays (1,2,4,8)/15. The latter yields each player the payoff 14/15. Hence, we see that the endogenous sequencing of the moves eliminates the mixed strategy equilibrium. The intuition is simple: If my opponent commits himself to the mixed equilibrium in the first stage, I have an incentive to wait since in that case I guarantee that we either coordinate on (T, L) or (B, R), i.e. I prevent a disequilibrium outcome. However, if I wait, my opponent does better by committing to his Stackelberg leader strategy.

Having eliminated the mixed equilibrium from Table 1a, let us now turn to the Pareto dominated pure strategy equilibrium "1,1". A glance at Table 1b shows that this is not as easily eliminated. Namely, B^1 and R^1 are undominated strategies in Table 1b so that (B^1, R^1) is a perfect equilibrium. Even stronger, (B^1, R^1) is a proper equilibrium (Myerson (1978)): For any $\varepsilon > 0$ that is sufficiently small the strategy pair in which each player plays the completely mixed strategy $(\varepsilon^3, \varepsilon^2, \varepsilon, 1 - \varepsilon - \varepsilon^2 - \varepsilon^3)$ is a 2ε-proper equilibrium in Table 1b. In fact, one may show that, in the game of Table 1b, the connected set of equilibria with payoff "1,1" contains a stable set as defined in Kohlberg and Mertens (1986). (Details are available from the authors upon request.) Intuitively, if players expect mistakes to occur with a relatively large probability in the second period, or if they believe that the unattractive mixed strategy equilibrium would be played in case the second period would be reached, then each player has a strong incentive to commit to the equilibrium "1,1" in the first period if he expects his opponent to do the same. Hence, it seems that Rosenthal's (1991) conclusion that only (T, L) makes sense in the game of Table 1a when aspects of commitment are taken into account was premature.

The concept of Nash equilibrium allows two interpretations. The first is as a necessary requirement for a self-enforcing agreement and the concepts of perfectness, properness and stability all correspond to this interpretation. Hence, from the preceeding paragraph we may conclude that an agreement on "1,1" is self-enforcing even when the timing of the moves is endogenous. An alternative interpretation of a Nash equilibrium is as a fixed point of an (unspecified) dynamic evolutionary or learning process. Several refinements have been proposed that correspond to this interpretation and it turns out that some of these allow the conclusion that players will coordinate on "2,2" when the timing of the moves is endogenous. Specifically, the concepts of persistent equilibria (Kalai and Samet (1984)) and curb*-equilibria (Basu and Weibull (1991)) force players to coordinate on (T, L). Let us illustrate this result for curb*-equilibria, i.e. for equilibria that belong to minimal sets of mixed strategy pairs that are closed with respect to taking undominated best responses.[1] The game of Table 1b has two sets that are closed under (undominated) best responses, viz. the entire strategy set and the set of strategies in which zero probability is assigned to B^1 and R^1. Obviously, only the latter set is minimal. Furthermore, each equilibrium in this set induces the outcome "2,2", hence, each curb* equilibrium of the game of Table 1b produces the commitment robust equilibrium of Table 1a. In Section 6 we show that the above argument generalizes to any game that admits a unique

[1] Since we here consider the reduced game, in this section this result holds also for curb-equilibria, i.e. equilibria that belong to minimal sets of mixed strategies that are closed with respect to taking best responses.

and pure commitment robust equilibrium. In particular, for each common interest game we, hence, have a "justification" for playing the Pareto-dominant equilibrium.

3 Commitment Robust Equilibria

We start our analysis from a given finite 2-person normal form game $g = (A_1, A_2, u_1, u_2)$. To avoid trivial cases we assume throughout the paper that $|A_i| \geq 2$. We write $S_i = \Delta(A_i)$ for the set of mixed strategies of player i in g and we denote a generic mixed strategy by s_i. We write $C(s_i) = \{a_i \in A_i : \ s_i(a_i) > 0\}$. The set of mixed strategy pairs is $S = S_1 \times S_2$, and, for $s \in S$, $C(s) = C(s_1) \times C(s_2)$. Throughout the paper we assume the mild regularity condition that different outcomes are associated with different payoffs, i.e.

$$\text{if } a \neq a', \text{then } u_i(a) \neq u_i(a') \quad (i \in \{1, 2\}) \tag{3.1}$$

Obviously, (3.1) is satisfied for generic games. This condition implies that the best reply against any pure strategy is unique. We denote the best reply of player j against a_i by $b(a_i)$ and write

$$u_i(a_i) = u_i(a_i, b(a_i)) \tag{3.2}$$

and

$$\bar{u}_i \quad = \quad \max_{a_i \in A_i} u_i(a_i). \tag{3.3}$$

Hence, $u_i(a_i)$ is player i's payoff when he commits to a_i and $\bar{u}_i$ is this player's payoff when he acts as the Stackelberg leader. Note that (3.1) implies that there is a unique strategy attaining the maximum in (3.3). We say that a player has an incentive to move first if he would be better off if he would be allowed to act as a Stackelberg leader. If no player prefers being a leader to playing simultaneously, the equilibrium is said to be commitment robust. Formally:

Definition 1 *(i) Player i has an incentive to move first at s if $u_i(s) < \bar{u}_i$.*

(ii) An equilibrium is commitment robust (is a CRE) if no player has an incentive to move first at s.

Note that any pure strategy Nash equilibrium s satisfies $u_i(s) \leq \bar{u}_i$ for $i = 1, 2$. Hence, if a pure equilibrium s is a CRE, then $u_i(s) = \bar{u}_i$ for $i = 1, 2$ and s is a subgame perfect equilibrium outcome in each of the two games where one of the players is forced to move first. Conversely, if s is pure and s is a subgame perfect equilibrium outcome of each of these two games, then s is a Nash equilibrium of g and no player has an incentive to move first at s. Obviously, any such equilibrium must Pareto-dominate any other pure Nash equilibrium and we have proved

Corollary 1 *A pure strategy equilibrium s is a CRE if and only if s is an SPE outcome of the two games in which one of the players is forced to move first. A pure CRE Pareto-dominates any other pure Nash equilibrium. Hence, two pure CRE are payoff equivalent, and, therefore, each game has at most one pure CRE.*

It is easy to construct games without any CRE, take the Battle of the Sexes, for example. On the other hand, the set of games with a pure CRE is certainly not of measure zero: All games that are close to the game from Table 1a have (T, L) as a pure CRE. Two classes of games that do admit CRE are the strictly competitive games (Friedman (1983)) and the common interest games (Aumann and Sorin (1989)). The game g is said to be *strictly competitive* if for all s, s' and $i \neq j$, if $u_i(s) \geq u_i(s')$, then $u_j(s) \leq u_j(s')$. The game g is a game with *common interest* if there exists s such that $u_i(s) \geq u_i(s')$ for all i and s'.

Lemma 1 *a) If g is strictly competitive and s is a Nash equilibrium of g, then s is a CRE.*
b) If g has common interest and s is the Pareto dominant Nash equilibrium of g, then s is a CRE.

Proof. The proof of b) is trivial, so we only prove a). Assume s is an equilibrium of a strictly competitive game g and $u_1(s) < \bar{u}_1$. Let $\bar{u}_1 = u_1(\bar{a}_1) = u_1(\bar{a}_1, \bar{a}_2)$. Then $u_2(\bar{a}_1, a_2) \leq u_2(\bar{a}_1, \bar{a}_2)$ for all a_2, hence, $u_1(\bar{a}_1, a_2) \geq u_1(\bar{a}_1, \bar{a}_2) > u_1(s)$ for all a_2. But then by playing $\bar{a}_1$ player 1 could guarantee himself more than the equilibrium payoff $u_1(s)$, which is impossible. □

Corollary 1 is no longer correct for mixed equilibria. For example, the zero-sum game of Matching Pennies has a unique (symmetric) mixed strategy equilibrium. No player has an incentive to move first at this equilibrium: each player is sure to lose the game if he is forced to disclose his action before the opponent. However, the game in which player i is forced to move first has no SPE with the same outcome as the mixed equilibrium of Matching Pennies. Furthermore, a pure CRE may be Pareto dominated by a mixed equilibrium: Add to Matching Pennies a third strategy for each player such that if both play this strategy both lose half a penny, but both players lose one penny if only one player chooses this strategy. In each Stackelberg outcome, both players choose the third strategy, but the resulting equilibrium is Pareto dominated by the original mixed equilibrium of Matching Pennies.

The above problems associated with mixed strategies can be avoided by considering the two games where one player is allowed, but not forced, to move first. Hence, in these games the timing of one player's decision is determined endogenously. Formally, in round 1 player i chooses between to commit to an action a_i or not. If he commits himself, then the commitment a_i is revealed to player j. If not, player j is informed that player i did not commit himself and players choose their actions simultaneously. The reader can now easily verify that s is a CRE if and only if each of these two games has a subgame perfect equilibrium with outcome s.

Corollary 2 *An equilibrium s is a CRE if and only if s is an SPE outcome in each of the two games in which one player is allowed to move first.*

The definition of commitment robustness given above differs from that given in Rosenthal (1991). The difference concerns the treatment of mixed strategies. Rosenthal defines an equilibrium s to be commitment robust if s is an SPE outcome of each of the two games in which one of the players moves first by committing to a mixed strategy. Specifically, Rosenthal assumes that the mixed strategy that the player commits to can be communicated. Hence, he works with what Schelling (1960, p. 185) calls "fractional commitments" and Schelling already points out that these may be more efficient than pure ones. (Compare our discussion of the extended matching pennies game in which a mixed strategy equilibrium Pareto dominates the pure strategy equilibrium.) Of course, having the opportunity to commit to a mixed action can never be worse than having the opportunity to commit to a pure action and, hence, any equilibrium that is commitment robust according to Rosenthal (which will be called an RCRE) is also a CRE. As we have seen, a CRE need not be an RCRE. However, the reader may easily verify that for each of the two classes of games considered in Lemma 1 any CRE is also an RCRE.

As Corollary 2 shows, there are three differences between our concept and that of Rosenthal. First, although players can commit to mixed actions in both settings, only Rosenthal allows them to communicate these mixtures. In our case, the actual outcome of the randomization has to be communicated. Secondly, we allow the players to commit themselves, but we do not force them. In Rosenthal's setup players are forced to commit themselves. Another difference between the two concepts is that our concept is characterized by auxiliary games, while Rosenthal uses such games directly in the definition of his concept. The following Proposition shows that, for games satisfying a regularity condition that is slightly different from (3.1), RCRE could have been similarly defined directly by means of a pair of inequalities. The regularity condition in question has first been introduced in Lemke and Howson (1964) and is also satisfied for generic games. For $s_j \in S_j$ write $B(s_j)$ for the set of pure best replies of player i against s_j and define the matrix u_{is_j} as the restriction of u_i to $B(s_j) \times C(s_j)$. The regularity condition that Lemke and Howson impose is

$$\text{rank } (u_{is_j}) \geq |B(s_j)| \text{ for all } j \in \{1, 2\} \text{ and all } s_j \in S_j. \tag{3.4}$$

Condition (3.4) states that there are no linear dependencies between the rows and the columns of the payoff matrices. It implies that the equilibria are isolated and that the number of equilibria is odd.

Proposition 1 *For a game g and a player i define u_i^+ by means of*

$$u_i^+ = \max_{s_i \in S_i} \max_{a_j \in B(s_i)} u_i(s_i, a_j).$$

Let s^* be a Nash equilibrium of g. If $u_i(s^*) \geq u_i^+$ for $i = 1, 2$, then s^* is an RCRE. If g satisfies (3.4), then s^* is an RCRE if and only if $u_i(s^*) \geq u_i^+$ for $i = 1, 2$.

Proof. See Appendix. □

If g does not satisfy (3.4), then an RCRE s need not satisfy the inequalities $u_i(s) \geq u_i^+$ $(i = 1, 2)$. The game from Table 2 provides an example. (The game does not satisfy (3.1) but the payoffs may be slightly perturbed to satisfy this condition. However, care should be taken that the third pure strategy of player 2 remains a best response to the equilibrium strategy of player 1.) In this game, the equilibrium s in which both players choose each of the first two pure strategies with probability 1/2 is an RCRE. Player 1's equilibrium payoff is 1 while $u_1^+ = 3$, hence, $u_1(s) < u_1^+$. We note that a Nash equilibrium s satisfying $u_i(s) = u_i^+$ $(i = 1, 2)$ is called a Stackelberg equilibrium (SE) in d'Aspremont and Gérard-Varet (1980). Hence, we have that each SE is an RCRE and that each RCRE is a CRE and that neither of the converses holds. We leave further discussion to Section 7.

	L		C		R	
T	2	0	0	2	3	1
B	0	2	2	0	3	1

Table 2.

In comparing our concept of commitment robustness to that of Rosenthal, we believe that the setting corresponding to our definition is more natural, however, we do not want to be dogmatic: In our main result, CRE may be replaced by RCRE. Perhaps more importantly, as we already argued in the introduction, both concepts are somewhat unsatisfactory since they are based on games with asymmetric commitment possibilities. We would want to give both players the possibility to commit themselves so that commitments could arise endogenously. Hence, we want to investigate which equilibria of the original game are still viable when the order of the moves is endogenous. To formally address this issue, we use the 2-stage game of action commitment that has been introduced in Hamilton and Slutsky (1990). In this game, which will be referred to as γ^2, each player can choose between committing to an action in period 1 or to wait till period 2. Formally, the rules are as follows

Stage 1: Simultaneously the players choose to commit to actions in A_1 and A_2, respectively, or to wait till stage 2.

Stage 2: Each player i who did not yet choose an action in A_i is informed about what action his opponent j took in stage 1. After having received this information, the player is required to choose an action in A_i with players moving simultaneously if both still have to make a choice.

Payoffs: The players' actions in γ^2 lead to a unique outcome in A. If $a \in A$ results, then player i's payoff in γ^2 is $u_i(a)$. Hence, there is no discounting, moving late does not entail any cost. Also, there is no cost involved in moving early.

Hence, the question to be addressed in the remaining sections is whether only commitment robust equilibria of g correspond to "sensible" equilibrium outcomes of γ^2. We conclude this section by introducing some notation with respect to the action commitment game.

We will use σ_i to denote a strategy of player i in γ^2. A strategy pair σ induces a probability distribution p^σ on $A_1 \times A_2$. Note that p^σ may involve nontrivial correlation. p^σ is called the *outcome* of σ, and we say that σ induces the outcome s if $p^\sigma = s$. Player i's payoff resulting from $\sigma = (\sigma_1, \sigma_2)$ is denoted by $U_i(\sigma)$. Hence, $U_i(\sigma) = \sum_a p^\sigma(a) u_i(a)$. The normal form of γ^2 will be denoted as g^2. Note that any pure strategy that tells player i to wait and that does not prescribe to best respond to a_j in case j committed to a_j in period 1 is (weakly) dominated in this normal form. It is convenient to introduce a shorthand notation for some pure strategies. If $a_i \in A_i$, then we write a_i^1 for the pure strategy "commit to a_i in period 1" while a_i^2 denotes the strategy "wait till period 2 and best respond if the opponent moves in period 1, otherwise play a_i".

4 Nonrobustness of Mixed Strategy Equilibria

The relevance of mixed strategy equilibria has frequently been questioned. In particular, the traditional, naive[2], interpretation according to which players consciously randomize their actions is intuitively not very appealing. The reason a player must randomize in equilibrium is only to keep the others from deviating; the player himself will not lose

[2]We do not consider here more sophisticated interpretations like mixed strategies as beliefs or as population frequencies.

if he deviates and instead uses any of the pure strategies in the support of the mixed strategy. Furthermore, unlike a pure equilibrium, a mixed strategy equilibrium typically is not ex post self-enforcing: Given the realizations of the actions, some player would like to deviate. Aspects of timing only compound these difficulties and as a result it is unlikely that a mixed strategy equilibrium is viable in the action commitment game. Namely, consider an equilibrium s^* of a game g satisfying (3.4) in which at least one of the players randomizes. Then each player randomizes and there is positive probability that the realized actions do not constitute an equilibrium ex post. Hence, if player j plays s_j^* in period 1 of γ^2, then player i strictly prefers waiting to moving in the first period, and, consequently, both players playing s^* in the first period is not a Nash equilibrium of γ^2. Obviously, we also have that both players waiting till the second period of γ^2 and then playing s^* is a subgame perfect equilibrium only if s^* is commitment robust. In the next proposition we show that the latter is actually a necessary and sufficient condition for the existence of an SPE of γ^2 with outcome s^*. Hence, only a very limited set of mixed strategy equilibria is viable when the timing of the moves is endogenous and the following Proposition can be viewed as a foundation and formalization of the critique of mixed strategy Nash equilibria in the applied literature (cf. Bester (1992)). Since the proof of this Proposition is rather tedious (mainly because of the fact that one has to deal with the possibility that both players play with positive probability in both periods), it is relegated to the Appendix.

Proposition 2 *Let g be a game that satisfies the regularity condition (3.4) and let s^* be an equilibrium of g that is not in pure strategies. Then there exists a subgame perfect equilibrium σ of γ^2 with outcome s^* if and only if no player has an incentive to move first at s^*.*

5 Robustness of Pure Strategy Equilibria

In this section we show that for pure strategy equilibria of g the situation is fundamentally different from that for mixed equilibria: Each pure equilibrium is a subgame perfect equilibrium outcome of the action commitment game, and each pure equilibrium corresponds to a perfect equilibrium of the strategic form of the game. Specifically, if $\bar{a}$ is a pure equilibrium of g and if my opponent j decides to commit to $\bar{a}_j$ in γ^2, I can do no better but to commit to $\bar{a}_i$ as well.

Proposition 3 *If $\bar{a}$ is a pure strategy equilibrium of g, then the strategy profile $\sigma = (\sigma_1, \sigma_2)$, where σ_i tells player i to play a best response in stage 2-subgames where he is the only one to decide and to play $\bar{a}_i$ in all other subgames, is a subgame perfect equilibrium of γ^2.*

Proof. Trivial. □

To see whether some of the equilibria from Proposition 3 can be eliminated, we can make use of equilibrium refinements. As most of these refinements have been defined in the strategic form of the game, we take a normal form perspective in the rest of this paper. The next proposition shows that all pure equilibria from Proposition 3 satisfy the weakest of these more refined criteria, i.e. they are perfect equilibria.

Proposition 4 *If $\bar{a}$ is a pure equilibrium of g then the strategy profile $\sigma = (\sigma_1, \sigma_2)$ defined by $\sigma_i = \bar{a}_i^1$ (i.e. each player i commits to $\bar{a}_i$ in period 1) is a perfect equilibrium of g^2.*

Proof. It is obvious that σ is a Nash equilibrium of g^2. We will show that σ_i is an undominated strategy in g^2. This is sufficient to establish the result since in a 2-person game each equilibrium in undominated strategies is perfect (Van Damme (1987, Theorem 3.2.2.)). Since $\bar{a}_i$ is undominated in g and g satisfies (3.1), $\bar{a}_i$ can be dominated in g^2 only if it is dominated by a strategy in which player i moves only in period 2. Suppose this strategy tells player i to play the mixed action s_i in the second period in case the other player also waited. Let $a_j, a'_j \in A_j$ and consider a strategy for player j that tells him to wait till period 2 and then to play a_j if player i committed to $\bar{a}_i$, and to play a'_j

otherwise. It follows that $u_i(\bar{a}_i, a_j) \le u_i(s_i, a'_j)$ for all $a_j, a'_j \in A_j$. Since each player has a unique best reply against each pure strategy, substitution of $a_j = a'_j = \bar{a}_j$ implies that $s_i = \bar{a}_i$. Hence, $u_i(\bar{a}_i, a_j) = u_i(\bar{a})$ for all $a_j \in A_j$. This contradicts (3.1). □

At this stage it is appropriate to compare our work with Hamilton and Slutsky (1990). Hamilton and Slutsky consider the game g^2 where g is the standard quantity setting Cournot duopoly game. In their Theorem VIII they claim

> "the two Stackelberg equilibria are the only pure strategy equilibria in undominated strategies. Playing the Cournot equilibrium strategy at the first turn is dominated by waiting to play after one's rival."

This claim obviously is inconsistent with Proposition 4. Now it is certainly true that the standard Cournot duopoly game does not fit our context and discretized versions of this game need not satisfy (3.1). Nevertheless, it is not difficult to see that the above claim is wrong. Playing the Cournot equilibrium strategy at the first stage would indeed be dominated if players could be sure that they would continue with the Cournot equilibrium in the second stage in case both players still have to move, and if each player could be sure that the opponent would always best respond to a unilateral commitment. However, players cannot be sure of this since the logic of the perfectness concept forces them to consider the possibility of mistakes in the second period. The total quantity in the second period might be above the Cournot quantity, if only by mistake, and in this case it pays to commit to the Cournot quantity if the opponent does the same. It is the impression of the present authors that, although not stating it explicitly, Hamilton and Slutsky actually had this truncated game (that excludes second period mistakes) in mind when making the above claim. (See also their companion paper Hamilton and Slutsky (1993) in which in the proof of Theorem III they display the payoff matrix of the truncated game rather than of the full game g^2.)

Although the relevance of the truncated game may be questioned — after all, what is the rationale for allowing mistakes in the first period of γ^2 but excluding them in the second period? — we can state a result about it that generalizes the observation of Hamilton and Slutsky about the quantity setting duopoly game: An equilibrium s of the original game is an outcome of a perfect equilibrium of this truncated game if and only if it is a CRE. If s is pure, the proof of this result is straightforward. In the case s is in mixed strategies, the proof follows the same lines as that of Proposition 2. Formally, if $s^* \in S$, write $\gamma^2(s^*)$ for the extensive form game that results from γ^2 if we force each player to best respond in period 2 to a unilateral commitment of the opponent in period 1 and if we replace the subgame w at stage 2 in which both players still have to move by an endpoint with payoff vector $(u_1(s^*), u_2(s^*))$. Let $g^2(s^*)$ be the associated normal form. We have

Proposition 5 *s^* is a commitment robust equilibrium if and only if there exist a perfect equilibrium σ of $g^2(s^*)$ with outcome s^*.*

Proof. See Proposition 4 in Van Damme and Hurkens (1993). □

6 Endogenous Timing and Coordination

In this section we investigate whether more refined equilibrium notions allow one to draw the conclusion that only commitment robust equilibria are viable when the order of the moves is endogenous. Specifically, we address the question of whether in a game that has a unique and pure CRE, endogenous timing will force the players to coordinate on this CRE. We will show that some set-valued "evolutionary" concepts, viz. the notions of persistent equilibria (Kalai and Samet (1984)) and of curb* equilibria (Basu and Weibull (1991)) do indeed allow this conclusion. We first formally define these concepts. Write $\mathcal{B}(s_i)$ for the set of all mixed best replies against s_i hence $\mathcal{B}(s_i) = \Delta B(s_i)$ and $\mathcal{B}(s) = \mathcal{B}(s_2) \times \mathcal{B}(s_1)$. For a set $\tilde{S}$ of mixed strategy pairs, write $\mathcal{B}(\tilde{S}) = \cup_{s \in \tilde{S}} \mathcal{B}(s)$. Similarly, define $\mathcal{B}^*(\tilde{S})$ for the set of all undominated best replies against $\tilde{S}$, i.e. best replies that are undominated strategies.

Definition 2 *Let g be a 2-person normal form game.*

(i) A retract is a set $R = R_1 \times R_2$ where R_i is a nonempty, closed and convex subset of S_i.

(ii) Given a certain property (X) of retracts, an X-retract is a retract R that is minimal with respect to this property, i.e. there does not exist a retract R' with $R' \subset R$, $R' \neq R$, having property (X).

(iii) An X-equilibrium is an equilibrium that belongs to an X-retract.

(iv) Persistent retracts and curb retracts are defined, respectively, by the properties (P) and (C^*):*

$$\mathcal{B}(s) \cap R \neq \emptyset \quad \text{for all } s \text{ in some open neighborhood } \mathcal{O} \text{ of } R \tag{P}$$

$$\mathcal{B}^*(R) \subset R \tag{C^*}$$

We will make use of the following proposition.

Proposition 6 *(i) Every curb* retract contains a persistent retract.*

(ii) Every game has at least one curb, resp. persistent equilibrium.*

(iii) If R is a curb retract, then $R_i = \Delta(A_i')$ for some $A_i' \subset A_i$.*

(iv) Different curb retracts are disjoint.*

(v) If no player i has equivalent strategies[3] among his pure undominated ones in the game, then the properties (iii) and (iv) also hold for persistent retracts, i.e. each persistent retract is a convex hull of pure strategies and different persistent retracts are disjoint.

[3]Two pure strategies a_i, a_i' of player i are equivalent if $u_i(a_i, a_j) = u_i(a_i', a_j)$ for all pure strategies a_j of player j.

(vi) If the game satisfies (3.4), then a retract is curb if and only if it is persistent.*

Proof. The statements about persistence were already proved by Kalai and Samet (1984). The other statements were proved by Balkenborg (1992); we will just sketch the proofs of the latter statements.

A retract with property (C^*) also satisfies (P). (This proves (i).) Suppose that one of the extreme points of a retract with property (C^*) is not a pure strategy. The retract that is obtained by deleting all strategies that put positive weight on this mixed strategy also has property (C^*). (This proves (iii).) The existence of curb* retracts follows from (iii) and the fact that the whole strategy space has property (C^*). ((Combining this with (i) and the well known existence result for persistent equilibria proves (ii).) The intersection of two retracts with property (C^*) has again this property, or it is empty. (This proves (iv).) The proof of (vi) follows from (v) (in a game satisfying (3.4) there are no equivalent undominated strategies) together with the observation that, if $a_i \in B(s_j)$ then there exists s'_j close to s_j such that $\{a_i\} = B(s'_j)$, cf. the proof of Proposition 1. □

Before we come to the main result of this paper, we state

Lemma 2 *In g^2 no player has equivalent strategies among his pure undominated ones.*

Proof. Suppose on the contrary that player i has equivalent strategies among his pure undominated ones in g^2. Hence, there exist $a_i, \bar{a}_i \in A_i$ such that either (i) a_i^1 is equivalent with $\bar{a}_i^1$, or (ii) a_i^1 is equivalent with $\bar{a}_i^2$ or (iii) a_i^2 is equivalent with $\bar{a}_i^2$. In all cases it follows that $u_i(a_i, a_j) = u_i(\bar{a}_i, a'_j)$, for all $a_j, a'_j \in A_j$. This implies that $u_i(a_i, a_j) = u_i(a_i, a'_j)$, for all $a_j, a'_j \in A_j$. But this contradicts (3.1). □

The main result of this paper is[4]

Proposition 7 *Assume $\bar{a}$ is a pure CRE of g. Then*

[4]Since any RCRE is a CRE, this Proposition remains valid when "CRE" is replaced by "RCRE".

(i) Any curb, resp. persistent equilibrium of g^2 yields each player i a payoff of at least $u_i(\bar{a})$.*

(ii) If $\bar{a}$ is the unique CRE of g, then each curb, resp. persistent equilibrium of g^2 results in the outcome $\bar{a}$.*

Proof. In this proof, let "x" stand for "curb*" or "persistent". Lemma 2 implies that no player i has equivalent strategies among his pure undominated ones in g^2, hence x-retracts are convex hulls of pure undominated strategies and different x-retracts are disjoint (Proposition 6). Let the retract $\bar{R}$ be defined by

$$\bar{A}_i = \{\bar{a}_i^1\} \cup \{a_i^2 : \ a_i \in A_i\}, \ \bar{R}_i = \Delta \bar{A}_i, \ \bar{R} = \bar{R}_1 \times \bar{R}_2.$$

Note that $\bar{R}$ satisfies the properties (C^*) and (P). We will show that any x-retract is contained in $\bar{R}$.

The proof is easy in case some player i has a dominant strategy $\tilde{a}_i$ in g. Then $\tilde{a}_i = \bar{a}_i$ and we have that for all $a_i \neq \bar{a}_i$, a_i^2 is dominated by $\bar{a}_i^2$. Moreover, for all $a_j \in A_j$ and all $a_i \neq \bar{a}_i$ we have $U_i(\bar{a}_i^1, a_j^1) > U_i(a_i^1, a_j^1)$ (since $\bar{a}_i$ is dominant in g) and $U_i(\bar{a}_i^1, a_j^2) > U_i(a_i^1, a_j^2)$ (since $\bar{a}_i$ is player i's unique Stackelberg leader strategy). This implies that if R is an x-retract, then

$$R \subset \Delta(\{\bar{a}_i^1, \bar{a}_i^2\}) \times \Delta(\bar{A}_j) \subset \bar{R}.$$

Now assume that no player has a dominant strategy in g. Then it is easily seen that $\bar{a}_i^2$ is an undominated strategy for each player i. Furthermore, for each j there exists some $\tilde{a}_j$ such that $\bar{a}_i \not\in B(\tilde{a}_j)$. We will show that, if $a_j \neq \bar{a}_j$, then committing to a_j cannot belong to any x-retract. Note that if R is an x-retract and $a_j^1 \in R_j$, then $\bar{a}_i^2 \in R_i$. Namely, $\bar{a}_i^2$ is an undominated best response against a_j^1 (hence $\bar{a}_i^2 \in R_i$ if x stands for curb*, and $\bar{a}_i^2$ is the unique best response against $(1-2\varepsilon)a_j^1 + \varepsilon\tilde{a}_j^1 + \varepsilon\bar{a}_j^2$ (so that $\bar{a}_i^2 \in R_i$ if x stands for persistent). Now, since $\bar{a}_j^2$ is the unique best response against $(1-\varepsilon)\bar{a}_i^2 + \varepsilon\tilde{a}_i^1$ it follows that $(\bar{a}_1^2, \bar{a}_2^2) \in R$ whenever $a_i^1 \in R_i$ for some $a_i \neq \bar{a}_i$. Since $\bar{R}$ satisfies the properties (C^*) and (P), we have that any x-retract that contains $(\bar{a}_1^2, \bar{a}_2^2)$ is contained in $\bar{R}$ and there does not exist an x-retract containing some a_i^1 with $a_i \neq \bar{a}_i$. Hence, any x-retract is contained in $\bar{R}$.

Now note that, if players are restricted to choose strategies from $\bar{R}$, then each player i can guarantee the payoff $u_i(\bar{a})$ by playing $\bar{a}_i^1$. Consequently, if σ is an x-equilibrium of g^2, then $C(\sigma) \subset \bar{R}$ and $u_i(\sigma) \geq u_i(\bar{a})$ for $i = 1, 2$, which proves the first part of the proposition.

Now assume that there exists an x-equilibrium σ that results in an outcome different from $\bar{a}$. Then for each player i we must have $\sigma_i(\bar{a}_i^1) < 1$. Define the mixed strategy s_i of player i in g by

$$s_i(a_i) = (1 - \sigma_i(\bar{a}_i^1))^{-1} \sigma_i(a_i^2) \qquad (a_i \in A_i)$$

Since σ is an equilibrium of g^2 we have that s is an equilibrium of g and, furthermore $u_i(s) \geq \bar{u}_i$ for $i = 1, 2$, since each player i can guarantee $\bar{u}_i$ by playing $\bar{a}_i^1$. Hence, s is a CRE of g. This completes the proof. □

7 Conclusion and Related Literature

We have addressed the question of whether only equilibria at which no player has an incentive to move first are viable when the order of the moves is endogenously determined and players have the opportunity to commit themselves. We have seen that in order to answer this question in the affirmative one needs quite strong equilibrium concepts. However, if one is willing to accept such concepts, one can indeed conclude that endogenous timing forces players to coordinate on a unique and pure commitment robust equilibrium. We have restricted ourselves in this paper to 2-person games and we have only allowed one point in time at which a player can commit himself. It is important to investigate the extent to which our results depend on these assumptions. One can easily define the game γ^t in which there are $t - 1$ periods in which a player has the opportunity to commit. (Obviously $\gamma^1 = g$.) The reader can verify that our main results remain valid in this extended context. Hence, mixed equilibria are typically not viable and any curb* (resp. persistent) equilibrium of the game in which the players have $t-1$ opportunities to commit themselves results in the commitment robust equilibrium. We have not investigated whether our results extend to games with more than two players, although we expect they do. It is clear, however, that in some cases (as in the proof of Proposition 2) different techniques are needed.

For convenience we assumed throughout the paper regularity condition (3.1) to hold. It is worthwhile to remark that Propositions 3, 4 and 5 also hold under the milder regularity assumptions

$$u_i(a) = u_i(a') \text{ if and only if } u_j(a) = u_j(a') \quad (i \neq j) \tag{7.1}$$

and

$$|B(a_i)| = 1 \text{ for all } a_i \in A_i \quad (i = 1, 2) \tag{7.2}$$

However, for Proposition 7 to hold we then need the additional assumption that players have a unique Stackelberg leader strategy. To see why the latter assumption (which is also implied by (3.1)) is essential, consider game g in Table 3. Note that this game satisfies (7.1) and (7.2). (T, L) is a CRE in this game, however, the unique curb* (or persistent) retract of g^2 contains all first period commitment strategies, so that, in particular (B^1, R^1) is a curb* equilibrium.

	L		C		R	
T	3	3	0	0	0	0
M	2	2	3	3	0	0
B	0	0	4	1	2	2

Table 3.

It has to be admitted that the class of games with a commitment robust equilibrium, i.e. the class for which we were able to determine the outcome with endogenous timing in this paper is quite limited. In a companion paper (Van Damme and Hurkens (1994b)) we address the question of which outcomes can be expected for some economic games without a commitment robust equilibrium. One conclusion that we derive is that, if one of the Stackelberg equilibria of g Pareto dominates the other, then players will coordinate on this Pareto dominant Stackelberg equilibrium, provided that there are at least two time periods at which players can commit themselves. However, if the two Stackelberg equilibria are Pareto incomparable, new issues arise and the outcome may depend on the length of the timing game. For further details, the reader is referred to our companion paper.

We conclude with discussing some related papers. We refrain from discussing the (rather extensive) literature that is built on the idea that, if the equilibrium associated with a certain timing of the moves Pareto dominates all equilibria associated with any alternative sequencing, then the Pareto optimal sequencing will result. (See the literature that is cited in Hamilton and Slutsky (1990) in relation with "the extended game with observable delay".) We have already discussed Rosenthal (1991) and Hamilton and Slutsky (1990) in the Sections 3 and 5. In Hamilton and Slutsky (1993), the action commitment game is studied for the special case where g is a 2×2 game in which exactly one player (say player 1) has a dominant strategy. Obviously, g has a unique equilibrium s^* in this case. Hamilton and Slutsky consider the reduced game $g^2(s^*)$ introduced in Section 5 and they show that this has a unique perfect equilibrium. By the above Proposition 7 the outcome of this equilibrium is s^* when s^* is a CRE. If s^* is not a CRE, then the outcome is that player 1 commits to his dominated strategy. d'Aspremont and Gérard-Varet (1980) define a Nash equilibrium s to be a Stackelberg equilibrium if $u_i(s) = u_i^+$ for $i = 1, 2$. As we have already seen, a Stackelberg equilibrium is an RCRE, but the converse need not hold. They point out that in a strictly competitive game, each Nash equilibrium is a Stackelberg equilibrium. (Cf. Lemma 1, where we actually proved this slightly stronger result.) They also show that each "twisted equilibrium" that is Pareto optimal is a Stackelberg equilibrium. (s is a twisted equilibrium if $u_i(s) \leq u_i(s_i, s_j')$ for all s_j' and all $i \neq j$, hence, if player i does not lose when j deviates.)

d'Aspremont and Gérard-Varet do not discuss endogenous timing of the moves. Such aspects are discussed in two papers by Robson. Robson (1989) considers a 2-stage game that is closely related to the action commitment game that has been analyzed in this paper. The two differences are that mixed strategies can be communicated and that moving early entails a cost. Specifically, if the outcome s results and i moves in period t ($t \in \{1, 2\}$), then player i's payoff is equal to $u_i = u_i(s) - (2-t)c_i$ where $c_i > 0$. Robson's most important result is that, if each Nash equilibrium is a Stackelberg equilibrium (as defined in d'Aspremont and Gérard-Varet (1980)), then all pure subgame perfect equilibria of the timing game involve choosing a Stackelberg equilibrium. (The result follows easily from the observation that there cannot be a pure equilibrium in which both players move early. It is worthwhile to point out that this result depends essentially on the restriction to pure strategies. Namely, consider the game from Table 1a and assume that moving in period 1 costs $\varepsilon > 0$. Now (B^1, R^1) is no longer an equilibrium, but there is a mixed equilibrium close to it, namely each player choosing $(0, \varepsilon, 2\varepsilon, 1-3\varepsilon)$. Hence, in this equilibrium each player commits immediately to "1,1" with probability $1-3\varepsilon$; if the game reaches the second stage without any commitments being made, players play the mixed equilibrium of the underlying game. Robson (1990) generalizes the model from Robson (1989) by allowing for more periods in which the players can commit themselves. It is assumed that the earlier the player moves, the costlier it is and it is shown that under certain conditions the timing game has a unique pure subgame perfect equilibrium. The equilibrium outcome is that the player who gains most from leading indeed becomes the leader. Again the result depends crucially on the restriction to pure strategies. Another drawback of Robson's model is that the cost functions are specified exogenously, i.e. it is not explained why there would be a cost of moving early. For the special class of common interest games, Balkenborg (1993) has shown that Proposition 7 continues to hold for another set-valued evolutionary concept, viz. that of direct evolutionary stable sets, abbreviated DESS. Balkenborg shows that a set E of mixed strategy vectors is a DESS if and only if

$$E = \cup_{\sigma \in E}(\mathcal{B}(\sigma_2) \times \{\sigma_2\}) = \cup_{\sigma \in E}(\mathcal{B}(\sigma_1) \times \{\sigma_1\})$$

and he proves that a DESS of g^2 exists if and only if g is a common interest game.

Furthermore, in case g has common interest, there is a unique DESS and it consists of all strategy combinations yielding the Pareto efficient payoff vector.

In the game g^2 it is assumed that, if a player commits himself, this commitment is perfectly observed by the opponent. Bagwell (1992) considers the case where only one player can commit himself but where this commitment is only imperfectly observed and he claims that in this case the strategic benefits of the commitment possibility evaporate. As we show in Van Damme and Hurkens (1994a), Bagwell's result depends essentially on the restriction to pure strategies: When mixed strategies are taken into account, there is continuity in the sense that all "sensible" equilibria of the noisy game converge to the Stackelberg equilibrium of the game without noise. We believe that a similar continuity result can be established in the present setting with two-sided commitment possibilities.

Appendix

This Appendix provides the proofs of Proposition 1 and 2.

Proof of Proposition 1. Consider the game γ^{ij} in which player i acts as a Stackelberg leader, with this player's mixed action being revealed to player j. Because of the bilinearity of u_i, player i's payoff if he chooses s_i and j best responds, is at most $\max_{a_j \in B(s_i)} u_i(s_i, a_j)$. Hence, each SPE yields player i at most u_i^+. If $u_i(s^*) = u_i^+$, then any strategy pair σ with $\sigma_i = s_i^*$ and σ_j with $\sigma_j(s_i^*) = s_j^*$ and $\sigma_j(s_i) \in B(s_i)$ for all s_i is an SPE of γ^{ij}. This proves the first part of the proposition.

To prove the second part, let $s_i^+ \in S_i$ and $a_j^+ \in B(s_i^+)$ be such that $u_i(s_i^+, a_j^+) = u_i^+$. Because of (3.4), there exists $x \in \mathbb{R}^{A_i}$ with $x(a_i) = 0$ if $a_i \not\in C(s_i^+)$ such that $u_j(x, a_j^+) > 0$ and $u_j(x, a_j) = 0$ for all $a_j \in B(s_i^+) \backslash \{a_j^+\}$. (Here $u_j(x, a_j)$ is shorthand notation for $\sum_{a_i} x(a_i) u_j(a_i, a_j)$.) Write $t = \sum_{a_i} x(a_i)$ and $s_i(\varepsilon, t) = (1 + \varepsilon t)^{-1}(s_i^+ + \varepsilon x)$. Then $s_i(\varepsilon, t) \in S_i$ if ε is sufficiently small, $B(s_i(\varepsilon, t)) = \{a_i^+\}$ and $s_i(\varepsilon, t) \to s_i^+$ as $\varepsilon \to 0$. Hence, if s is an SPE outcome of γ^{ij} we must have $u_i(s) \geq u_i(s_i(\varepsilon, t), a_j^+)$, and, therefore, $u_i(s) \geq u_i(s_i^+, a_j^+) = u_i^+$. This completes the proof of the second part of the proposition.

□

Before we give the proof of Proposition 2, it will be convenient to first state a Lemma that lists some properties of games satisfying (3.4).

Lemma 3 *Let g satisfy (3.4) and let s be an equilibrium of g that is not pure. Then*

(i) $|C(s_1)| = |C(s_2)| = |B(s_1)| = |B(s_2)|$

(ii) If s' is an equilibrium and $C(s') = C(s)$, then $s' = s$.

(iii) For each $a_i \in C(s_i)$, there exists $a_j \in C(s_j)$ such that $a_j \notin B(a_i)$.

(iv) For each $a_i \in C(s_i)$, there exists $a_j \in C(s_j)$ such that $a_i \notin B(a_j)$.

Proof.

(i) Since s is an equilibrium, we have $C(s_i) \subset B(s_j)$ for $i, j \in \{1, 2\}$. Since the rank of a matrix cannot exceed the number of columns, (3.4) implies $|B(s_j)| \leq |C(s_j)|$ for all s_j. Combining these observations yields $|C(s_1)| \leq |B(s_2)| \leq |C(s_2)| \leq |B(s_1)| \leq |C(s_1)|$, hence, all inequalities must be equalities.

(ii) Assume s and s' are equilibria with $C(s) = C(s')$. If $u_i(s) = 0$, then the columns of the matrix u_{is_j} are dependent, hence, (3.4) is violated. So assume $u_i(s) \neq 0$. Consider the vector v with a_j–th coordinate equal to $v(a_j) = u_i(s')s_j(a_j) - u_i(s)s'_j(a_j)$. Then, if we premultiply v by u_{is_j} we get zero, hence by (3.4) v must be the zero vector. Therefore, $u_i(s')s_j = u_i(s)s'_j$ and since both s_j and s'_j are probability vectors $u_i(s') = u_i(s)$. But then $s_j = s'_j$. A similar argument implies that $s_i = s'_i$.

(iii) This follows immediately from the fact that $|B(a_i)| = 1$ and $|C(s_j)| \geq 2$.

(iv) If $a_i \in B(a_j)$ for all $a_j \in C(s_j)$, then $\{a_i\} = B(a_j)$ for all such a_j, hence $\{a_i\} = B(s_j)$ contradicting (i). □

Before we come to the proof of Proposition 2 we need to introduce some notation.

Assumption (3.1) forces player j, in any SPE, to play $b(a_i)$ in the second period when player i committed himself unilaterally to a_i. Hence, an SPE strategy σ_i of player i only needs to tell which action from $A_i \cup \{w_i\}$ to choose in stage 1 and what to do in stage 2 in the information set $w = (w_1, w_2)$ that corresponds to the case where both players waited. For such a strategy, we write $\sigma_i = (\sigma_i^1, \sigma_i^2)$ where σ_i^1 denotes the randomization at time 1 and σ_i^2 is the mixed action in information set w. For the case where $\sigma_i^1(w_i) < 1$, i.e. player i moves with positive probability in the first period, it will also be convenient to write

$$\omega_i = \sigma_i^1(w_i), \;\; s_i^1 = (1-\omega_i)^{-1}\sigma_i^1, \;\text{ and } s_i^2 = \sigma_i^2 \tag{A.1}$$

hence, s_i^1 is the mixed action that player i plays if he moves in period 1. In this case we will also write $\sigma_i = (\omega_i, s_i^1, s_i^2)$. Writing

$$\delta_i(a) = \begin{cases} 1 & \text{if } \; a_i = b(a_j) \\ 0 & \text{otherwise,} \end{cases} \tag{A.2}$$

we see that the outcome p^σ of an SPE σ is given by

$$p^\sigma(a) = \sigma^1(a) + (1-\omega_1)\omega_2\sigma_1^1(a_1)\delta_2(a) + \omega_1(1-\omega_2)\sigma_2^1(a_2)\delta_1(a) + \omega_1\omega_2\sigma^2(a), \tag{A.3}$$

where we have used $\sigma^t(a)$ as a shorthand notation for $\sigma_1^t(a_1)\sigma_2^t(a_2)$.

Proof of Proposition 2.

Sufficiency. No player has an incentive to move first at s^* if and only if $u_i(s^*) \geq \bar{u}_i$ for $i = 1, 2$. In this case the strategy pair $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_i^1 = w_i$, $\sigma_i^2 = s_i^*$ is a subgame perfect equilibrium of γ^2 with outcome s^*.

Necessity. Assume $u_i(s^*) < \bar{u}_i$ for some i and that σ is a subgame perfect equilibrium of γ^2 with outcome s^*. We will derive a contradiction. Without loss of generality we may assume $u_1(s^*) < \bar{u}_1$. The proof is divided into a number of steps.

Step 1. $\omega_i = \sigma_i^1(w_i) < 1$ for $i = 1, 2$, hence, each player moves with positive probability in the first period.

Clearly, we must have $\sigma_2^1(w_2) < 1$, for otherwise player 1 can guarantee the payoff $\bar{u}_1$ by committing in the first period, hence $U_1(\sigma) \geq \bar{u}_1$ and σ cannot have the outcome s^*. Suppose that player 1 waits till the second period, i.e. $\sigma_1^1(w_1) = 1$. Let a^* be such that $\sigma_2^1(a_2^*) > 0$ and $a_1^* \in C(s_1^*) \backslash B(a_2^*)$. Then the outcome a^* can only be obtained if it is played in the second period, hence, $p^\sigma(a^*) \leq \omega_2 s^*(a^*)$, a contradiction.

Step 2. $\omega_i = \sigma_i^1(w_i) > 0$ for $i = 1, 2$, hence, with positive probability each player waits till the second period of γ^2 in σ.

Namely, assume $\sigma_i^1(w_i) = 0$ for some i. Then $\sigma_i^1(a_i) = s_i^*(a_i)$ for all a_i and i chooses at least two strategies with positive probability (Lemma 3). Furthermore, since there does not exist $a_j \in A_j$ with $a_j \in B(a_i)$ for all $a_i \in C(s_i^*)$, we have $U_j(\sigma_i, a_j^2) > U_j(\sigma_i, a_j^1)$ for all a_j, hence, $\sigma_j^1(w_j) = 1$. This contradicts step 1.

From the steps 1 and 2 we can conclude that the stage-2 subgame (w_1, w_2) is reached with positive probability if σ is played in γ^2. Since σ is a subgame perfect equilibrium of γ^2, σ^2 must be a Nash equilibrium of g. Furthermore, obviously $C(\sigma^2) \subset C(s^*)$. We will distinguish two cases:

Case A $C(\sigma^2) = C(s^*)$, hence (by Lemma 3) $\sigma^2 = s^*$.

Case B $C(\sigma^2) \neq C(s^*)$, hence (by Lemma 3), there exists for each player i a strategy a_i^* such that $a_i^* \in C(s_i^*) \backslash C(\sigma_i^2)$.

We first continue the proof for case A.

Step A3. For each player i: If $\sigma_i^1(a_i) > 0$, then $u_i(a_i) > u_i(s^*)$, hence, a player commits himself only to strategies that yield more than the equilibrium payoff.

Suppose, to the contrary, that there exists some $\bar{a}_i$ such that $\sigma_i^1(\bar{a}_i) > 0$ but $u_i(\bar{a}_i) \leq u_i(s^*)$. Since player i must be indifferent between $\bar{a}_i^1$ and a_i^2 for any $a_i \in C(s^*)$, we must have

$$\sum_{a_j} \sigma_j^1(a_j) u_i(\bar{a}_i, a_j) + \omega_j u_i(\bar{a}_i) = \sum_{a_j} \sigma_j^1(a_j) u_i(a_j) + \omega_j u_i(s^*)$$

hence

$$\begin{aligned} u_i(\bar{a}_i) &= u_i(s^*) \quad \text{and} \\ u_i(\bar{a}_i, a_j) &= u_i(a_j) \quad \text{for all } a_j \in C(\sigma_j^1). \end{aligned}$$

In particular, $\bar{a}_i$ must be a best response against any pure action that player j plays with positive probability in the first period. From Lemma 3 we can conclude that there exists $a_j^* \in C(s_j^*)$ such that $\bar{a}_i \not\in B(a_j^*)$ and $\sigma_j^1(a_j^*) = 0$. We may assume that $a_j^* \in B(\bar{a}_i)$, since otherwise by (A.3)

$$p^\sigma(\bar{a}_i, a_j^*) = \omega_1 \omega_2 s^*(\bar{a}_i, a_j^*) < s^*(\bar{a}_i, a_j^*).$$

Applying Lemma 3 once more we see that we can find a_i^* such that $a_j^* \not\in B(a_i^*)$. Then, for $a^* = (a_1^*, a_2^*)$,

$$p^\sigma(a^*) = \omega_1 \omega_2 s^*(a^*) < s^*(a^*),$$

another contradiction.

Step A4. Conclusion of the proof for case A.

We have seen up to now that each player must move in both periods with positive probability and that for each player i and for each action a_i with $\sigma_i^1(a_i) > 0$ we have $u_i(a_i) > u_i(s^*)$. Since $U_i(\sigma) = u_i(s^*)$ and since waiting is a best response for each player we have

$$\sum_{a_j} \sigma_j^1(a_j) u_i(a_j) + \omega_j u_i(s^*) = u_i(s^*)$$

so that there must exist a_j^* with $\sigma_j^1(a_j^*) > 0$ and $u_i(a_j^*) \leq u_i(s^*)$. Hence, $\sigma_j^1(a_j^*) > 0$ and $u_i(a_i, a_j^*) \leq u_i(s^*)$ for all $a_i \in A_i$, and, therefore

$$\sigma_j^1(a_j^*) > 0 \quad \text{and} \quad a_j^* \not\in B(a_i) \quad \text{for all } a_i \text{ with } \sigma_i^1(a_i) > 0.$$

Consider $a^* = (a_1^*, a_2^*)$. Then $p^\sigma(a^*) = \sigma^1(a^*) + \omega_1\omega_2 s^*(a^*)$, hence, since $p^\sigma = s^*$, we must have

$$\sigma^1(a^*) = (1 - \omega_1\omega_2) s^*(a^*). \tag{A.4}$$

On the other hand, for each player i, we have by summing up (A.3) over all a_j, $p^\sigma(a_i^*) = \sigma_i^1(a_i^*) + \omega_1\omega_2 s_i^*(a_i^*)$, hence

$$\sigma^1(a^*) = (1 - \omega_1\omega_2)^2 s^*(a^*). \tag{A.5}$$

Combining (A.4) with (A.5) and using $a^* \in C(s^*)$ yields $\omega_1\omega_2 \in \{0, 1\}$. But this contradicts the steps 1 and 2 and completes the proof for Case A.

We now continue with the proof of case B. Let a^* be such that, for each player i, $a_i^* \in C(s_i^*) \backslash C(\sigma_i^2)$.

Step B3. For all players i, j and all actions, if $a_j \in C(s_j^*)$ and $a_j \not\in B(a_i^*)$, then $a_i^* \in B(a_j)$.

Assume there exists $\bar{a}_j \in C(s_j^*)$ such that $\bar{a}_j \not\in B(a_i^*)$ and $a_i^* \not\in B(\bar{a}_j)$. In σ the outcome $(a_i^*, \bar{a}_j)$ can occur only if both players play it in the first period, hence

$$s^*(a_i^*, \bar{a}_j) = \sigma^1(a_i^*, \bar{a}_j). \tag{A.6}$$

By summing up (A.3) over all a_l it follows that, for $k \neq l$, $s_k^*(a_k) = p^\sigma(a_k) \geq \sigma_k^1(a_k)$. Combining this with (A.6) and (A.3) it follows that

$$\sigma_i^1(a_i^*) = s_i^*(a_i^*) \qquad \text{and} \qquad a_i^* \not\in B(a_j) \text{ if } \sigma_j^1(a_j) > 0.$$

This implies that any outcome (a_i^*, a_j) with $a_j \not\in B(a_i^*)$ can occur only in the first period, hence,

$$s^*(a_i^*, a_j) = \sigma^1(a_i^*, a_j) \qquad \text{for all } a_j \notin B(a_i^*),$$

and hence, $s_j^*(a_j) = \sigma^1(a_j)$ for $a_j \not\in B(a_i^*)$. This implies that if player j acts in period two, then he plays the unique $\bar{a}_j \in B(a_i^*)$ for sure. But this implies (by Lemma 3(i)) that $\bar{a}_j \in B(a_i)$ for all $a_i \in C(s_i^*)$. This contradicts Lemma 3(iv).

Step B4. Conclusion of the proof for case B.

Consider the pair $a^* = (a_1^*, a_2^*)$. Without loss of generality, step B3 allows us to assume that $a_1^* \in B(a_2^*)$. By Lemma 3 we have

$$\text{if } a_1 \in C(s_1^*),\ a_1 \neq a_1^*, \text{ then } a_1 \not\in B(a_2^*).$$

By step B3, therefore $a_2^* \in B(a_1)$ for all such a_1. But then we have that $a_2^* \in B(\sigma_1^2)$ and $a_2^* \not\in C(\sigma_2^2)$. By Lemma 3 this is impossible since σ^2 is an equilibrium of g and g satisfies the regularity condition (3.4). □

References

Aumann, R.J., and S. Sorin (1989). "Cooperation and Bounded Recall", *Games and Economic Behavior* **1**, 5-39.

Bagwell, K. (1992). "Commitment and Observability in Games". Northwestern University DP 1014.

Balkenborg, D. (1992). "The Properties of Persistent Retracts and Related Concepts", Ph.D. Thesis, University of Bonn.

Balkenborg, D. (1993). "A Note on Evolutionary Stability in Games with Commitment Possibilities", private communication.

Basu, K., and J. Weibull (1991). "Strategy Subsets Closed Under Rational Behavior", *Economics Letters* **36**, 141-146.

Bester, H. (1992). "Bertrand Equilibrium in a Differentiated Duopoly", *International Economic Review* **33**, 433-448.

Van Damme, E. (1987). *Stability and Perfection of Nash Equilibria*, Springer Verlag, Berlin.

Van Damme, E. and S. Hurkens (1993). "Commitment Robust Equilibria and Endogenous Timing". CentER Discussion Paper 9356.

Van Damme, E. and S. Hurkens (1994a). "Games with Imperfectly Observable Commitment". CentER, mimeo.

Van Damme, E. and S. Hurkens (1994b). "Equilibrium Selection in some Economic Games of Timing". CentER, in progress.

d'Aspremont, C. and L.-A. Gérard-Varet (1980). "Stackelberg-Solvable Games and Pre-Play Communication", *Journal of Economic Theory* **23**, 201-217.

Friedman, J.W. (1983). "On Characterizing Equilibrium Points in Two Person Strictly Competitive Games", *International Journal of Game Theory* **12**, 245-247.

Hamilton, J.H., and S.M. Slutsky (1990). "Endogenous Timing in Duopoly Games: Stackelberg or Cournot Equilibria", *Games and Economic Behavior* **2**, 29-46.

Hamilton, J.H., and S.M. Slutsky (1993). "Endogenizing the Order of Moves in Matrix Games", *Theory and Decision* **34**, 47-62.

Kalai, E., and D. Samet (1984). "Persistent Equilibria in Strategic Games", *International Journal of Game Theory* **13**, 129-144.

Kohlberg, E., and J.-F. Mertens (1986). "On the Strategic Stability of Equilibria", *Econometrica* **54**, 1003-1039.

Lemke, C.E., and J.T. Howson (1964). "Equilibrium Points of Bimatrix Games", *Journal of Social Industrial Applied Mathematics* **12**, 413-423.

Myerson, R. (1978). "Refinements of the Nash Equilibrium Concept", *International Journal of Game Theory* **7**, 73-80.

Robson, A. (1989). "On the Uniqueness of Endogenous Strategic Timing", *Canadian Journal of Economics* **22**, 917-921.

Robson, A. (1990). "Duopoly with Endogenous Strategic Timing: Stackelberg Regained", *International Economic Review* **31**, 263-274.

Rosenthal, R.W. (1991). "A Note on Robustness of Equilibria with Respect to Commitment Opportunities", *Games and Economic Behavior* **3**, 237-243.

Schelling, T.C. (1960). *The Strategy of Conflict*, Harvard University Press, Cambridge, Mass.

Von Stackelberg, H. (1934). *Marktform und Gleichgewicht*, Springer Verlag, Vienna and Berlin.